Derivations with identity predicate require additional inference rules.

\[
\begin{align*}
\frac{t_1 = t_2}{\varphi(t_1)} & \quad \text{=Intro} \\
\frac{t_1 = t_2}{\varphi(t_2)} & \quad \text{=Elim} \\
\frac{t_1 = t_2}{\varphi(t_1)} & \quad \text{=Elim}
\end{align*}
\]

In the above rules, \(t, t_1,\) and \(t_2\) are closed terms. The =Intro rule allows us to derive any identity statement of the form \(t = t\) outright, from no assumptions.

**Example ntd.1.** If \(s\) and \(t\) are closed terms, then \(\varphi(s), s = t \vdash \varphi(t):\)

\[
\frac{s = t \quad \varphi(s)}{\varphi(t)} \quad \text{=Elim}
\]

This may be familiar as the “principle of substitutability of identicals,” or Leibniz’ Law.

**Problem ntd.1.** Prove that = is both symmetric and transitive, i.e., give derivations of \(\forall x \forall y (x = y \rightarrow y = x)\) and \(\forall x \forall y \forall z ((x = y \land y = z) \rightarrow x = z)\)

**Example ntd.2.** We derive the sentence

\[
\forall x \forall y ((\varphi(x) \land \varphi(y)) \rightarrow x = y)
\]

from the sentence

\[
\exists x \forall y (\varphi(y) \rightarrow y = x)
\]

We develop the derivation backwards:

\[
\exists x \forall y (\varphi(y) \rightarrow y = x) \quad [\varphi(a) \land \varphi(b)]^1
\]

\[
\vdots
\]

\[
\frac{a = b}{((\varphi(a) \land \varphi(b)) \rightarrow a = b)} \quad \text{=Intro}
\]

\[
\frac{\forall y ((\varphi(a) \land \varphi(y)) \rightarrow a = y)}{\forall x \forall y ((\varphi(x) \land \varphi(y)) \rightarrow x = y)} \quad \text{=Intro}
\]

We’ll now have to use the main assumption: since it is an existential formula, we use \(\exists\text{Elim}\) to derive the intermediary conclusion \(a = b.\)
[\forall y (\varphi(y) \rightarrow y = c)]^2
[\varphi(a) \land \varphi(b)]^1

\[\forall y (\varphi(y) \rightarrow y = x)\]
\[a = b\]  \[\exists \text{Elim}\]

\[\forall y (\varphi(y) \rightarrow y = c)\]
\[\exists \text{Elim}\]
\[\forall \text{Intro}\]
\[\forall \text{Intro}\]
\[\forall \text{Intro}\]

The sub-derivation on the top right is completed by using its assumptions to show that \(a = c\) and \(b = c\). This requires two separate derivations. The derivation for \(a = c\) is as follows:

\[\forall y (\varphi(y) \rightarrow y = c)\]
\[\forall \text{Intro}\]
\[\forall \text{Intro}\]
\[\forall \text{Intro}\]

From \(a = c\) and \(b = c\) we derive \(a = b\) by \(\Rightarrow\text{Elim}\).

**Problem ntd.2.** Give derivations of the following formulas:

1. \(\forall x \forall y ((x = y \land \varphi(x)) \rightarrow \varphi(y))\)

2. \(\exists x \varphi(x) \land \forall y \forall z ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \rightarrow \exists x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x))\)

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**Bibliography**