Derivations with Identity predicate require additional inference rules.

\[
\begin{align*}
\frac{t = t}{\varphi(t)} & = \text{Intro} \\
\frac{t_1 = t_2}{\varphi(t_2)} & = \text{Elim} \\
\frac{t_1 = t_2}{\varphi(t_1)} & = \text{Elim}
\end{align*}
\]

In the above rules, \(t, t_1, t_2\) are closed terms. The \(=\text{Intro}\) rule allows us to derive any identity statement of the form \(t = t\) outright, from no assumptions.

**Example ntd.1.** If \(s\) and \(t\) are closed terms, then \(\varphi(s), s = t \vdash \varphi(t)\):

\[
\frac{s = t}{\varphi(s)} = \text{Elim}
\]

This may be familiar as the “principle of substitutability of identicals,” or Leibniz’ Law.

**Problem ntd.1.** Prove that \(=\) is both symmetric and transitive, i.e., give derivations of \(\forall x \forall y (x = y \rightarrow y = x)\) and \(\forall x \forall y \forall z ((x = y \land y = z) \rightarrow x = z)\)

**Example ntd.2.** We derive the sentence

\[
\forall x \forall y ((\varphi(x) \land \varphi(y)) \rightarrow x = y)
\]

from the sentence

\[
\exists x \forall y (\varphi(y) \rightarrow y = x)
\]

We develop the derivation backwards:

\[
\begin{align*}
\exists x \forall y (\varphi(y) \rightarrow y = x) & \quad [\varphi(a) \land \varphi(b)]^1 \\
& \quad \vdots \\
& \quad a = b \\
& \quad \frac{((\varphi(a) \land \varphi(b)) \rightarrow a = b)}{\forall y ((\varphi(a) \land \varphi(y)) \rightarrow a = y)} \quad \forall \text{Intro} \\
& \quad \frac{\forall x \forall y ((\varphi(x) \land \varphi(y)) \rightarrow x = y)}{\forall x \forall y ((\varphi(x) \land \varphi(y)) \rightarrow x = y)} \quad \forall \text{Intro}
\end{align*}
\]

We’ll now have to use the main assumption: since it is an existential formula, we use \(\exists\text{Elim}\) to derive the intermediary conclusion \(a = b\).

*identity rev: ad37848 (2024-05-01) by OLP / CC–BY*
The sub-derivation on the top right is completed by using its assumptions to show that $a = c$ and $b = c$. This requires two separate derivations. The derivation for $a = c$ is as follows:

\[
\exists x \forall y (\varphi(y) \to y = c)
\]

\[
\forall y (\varphi(y) \to y = c)
\]

\[
(\varphi(a) \land \varphi(b))
\]

\[
\varphi(a)
\]

\[
\varphi(a)
\]

\[
\exists x \forall y (\varphi(y) \to y = x)
\]

\[
a = b
\]

\[
\exists \text{Elim}
\]

\[
\forall \text{Intro}
\]

\[
\forall \text{Intro}
\]

\[
\forall \text{Intro}
\]

From $a = c$ and $b = c$ we derive $a = b$ by $=\text{Elim}$.

**Problem ntd.2.** Give derivations of the following formulas:

1. $\forall x \forall y ((x = y \land \varphi(x)) \to \varphi(y))$

2. $\exists x \varphi(x) \land \forall y \forall z ((\varphi(y) \land \varphi(z)) \to y = z) \to \exists x (\varphi(x) \land \forall y (\varphi(y) \to y = x))$

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**Bibliography**