ntd.1 Derivations with Identity predicate

fol:ntd:ide:sec

Derivations with identity predicate require additional inference rules.

$$\frac{}{t = t} = \text{Intro} \qquad \frac{t_1 = t_2 \qquad \varphi(t_1)}{\varphi(t_2)} = \text{Elim} \qquad \frac{t_1 = t_2 \qquad \varphi(t_2)}{\varphi(t_1)} = \text{Elim}$$

In the above rules, t, t_1, and t_2 are closed terms. The =Intro rule allows us to derive any identity statement of the form $t = t$ outright, from no assumptions.

Example ntd.1. If s and t are closed terms, then $\varphi(s), s = t \vdash \varphi(t)$:

$$\frac{s = t \qquad \varphi(s)}{\varphi(t)} = \text{Elim}$$

This may be familiar as the "principle of substitutability of identicals," or Leibniz' Law.

Problem ntd.1. Prove that = is both symmetric and transitive, i.e., give derivations of $\forall x \forall y\,(x = y \to y = x)$ and $\forall x \forall y \forall z((x = y \land y = z) \to x = z)$

Example ntd.2. We derive the sentence

$$\forall x \forall y\,((\varphi(x) \land \varphi(y)) \to x = y)$$

from the sentence

$$\exists x \forall y\,(\varphi(y) \to y = x)$$

We develop the derivation backwards:

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\exists x \forall y\,(\varphi(y) \to y = x) \quad [\varphi(a) \land \varphi(b)]^1 \\ \vdots \\ a = b\end{array}}{((\varphi(a) \land \varphi(b)) \to a = b)}\,{}^1 \to\text{Intro}}{\forall y\,((\varphi(a) \land \varphi(y)) \to a = y)}\,\forall\text{Intro}}{\forall x \forall y\,((\varphi(x) \land \varphi(y)) \to x = y)}\,\forall\text{Intro}$$

We'll now have to use the main assumption: since it is an existential formula, we use ∃Elim to derive the intermediary conclusion $a = b$.

$$
\dfrac{\dfrac{{}^{1}\dfrac{{}^{2}\dfrac{\exists x\,\forall y\,(\varphi(y)\to y=x) \qquad \begin{matrix}[\forall y\,(\varphi(y)\to y=c)]^2\\ [\varphi(a)\wedge\varphi(b)]^1\\ \vdots\\ a=b\end{matrix}}{a=b}\;\exists\text{Elim}}{((\varphi(a)\wedge\varphi(b))\to a=b)}\;{\to}\text{Intro}}{\forall y\,((\varphi(a)\wedge\varphi(y))\to a=y)}\;\forall\text{Intro}}{\forall x\,\forall y\,((\varphi(x)\wedge\varphi(y))\to x=y)}\;\forall\text{Intro}
$$

The sub-derivation on the top right is completed by using its assumptions to show that $a = c$ and $b = c$. This requies two separate derivations. The derivation for $a = c$ is as follows:

$$
\dfrac{\dfrac{[\forall y\,(\varphi(y)\to y=c)]^2}{\varphi(a)\to a=c}\;\forall\text{Elim} \qquad \dfrac{[\varphi(a)\wedge\varphi(b)]^1}{\varphi(a)}\;\wedge\text{Elim}}{a=c}\;{\to}\text{Elim}
$$

From $a = c$ and $b = c$ we derive $a = b$ by $=$Elim.

Problem ntd.2. Give derivations of the following formulas:

1. $\forall x\,\forall y\,((x = y \wedge \varphi(x)) \to \varphi(y))$
2. $\exists x\,\varphi(x) \wedge \forall y\,\forall z\,((\varphi(y) \wedge \varphi(z)) \to y = z) \to \exists x\,(\varphi(x) \wedge \forall y\,(\varphi(y) \to y = x))$

Photo Credits

Bibliography