Derivations in natural deduction are inductively generated from these: each
derivation either is an assumption on its own, or consists of one, two, or three
derivations followed by a correct inference.

**Definition ntd.1 (Derivation).** A *derivation* of a sentence \( \varphi \) from assumptions \( \Gamma \) is a tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in \( \Gamma \) or are discharged by an inference in the tree.
2. The bottommost sentence of the tree is \( \varphi \).
3. Every sentence in the tree except the sentence \( \varphi \) at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that \( \varphi \) is the *conclusion* of the derivation and that \( \varphi \) is *derivable* from \( \Gamma \).

**Example ntd.2.** Every assumption on its own is a derivation. So, e.g., \( \chi \) by itself is a derivation, and so is \( \theta \) by itself. We can obtain a new derivation from these by applying, say, the \( \land \) Intro rule,

\[
\frac{\varphi \quad \psi}{\varphi \land \psi} \land \text{Intro}
\]

These rules are meant to be general: we can replace the \( \varphi \) and \( \psi \) in it with any sentences, e.g., by \( \chi \) and \( \theta \). Then the conclusion would be \( \chi \land \theta \), and so

\[
\frac{\chi \quad \theta}{\chi \land \theta} \land \text{Intro}
\]

is a correct derivation. Of course, we can also switch the assumptions, so that \( \theta \) plays the role of \( \varphi \) and \( \chi \) that of \( \psi \). Thus,

\[
\frac{\theta \quad \chi}{\theta \land \chi} \land \text{Intro}
\]

is also a correct derivation.

We can now apply another rule, say, \( \rightarrow \) Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the antecedent of that conditional. So both of the following would be correct derivations:

\[
\frac{[\chi]^1 \quad \theta}{\chi \land \theta} \land \text{Intro} \quad \frac{\chi \land \theta}{\chi \rightarrow (\chi \land \theta)} \rightarrow \text{Intro}
\]

\[
\frac{[\theta]^1 \quad \theta}{\chi \land \theta} \land \text{Intro} \quad \frac{\theta \rightarrow (\chi \land \theta)}{\theta \land \chi} \land \text{Intro}
\]
Remember that discharging of assumptions is a permission, not a requirement: we don’t have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption $\varphi$ to be discharged:

$$
\frac{\psi}{\varphi \rightarrow \psi} \rightarrow \text{Intro}
$$

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Bibliography