ntd.1 Derivations

fol:ntd:der:sec We've said what an assumption is, and we've given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference. explanation

Definition ntd.1 (Derivation)**.** A *derivation* of a sentence φ from assumptions Γ is a tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in Γ or are discharged by an inference in the tree.
2. The bottommost sentence of the tree is φ.
3. Every sentence in the tree except φ is a premise of a correct application of am inference rule whose conclusion stands directly below that sentence in the tree.

We then say that φ is the *conclusion* of the derivation and that φ is *derivable* from Γ.

Example ntd.2. Every assumption on its own is a derivation. So, e.g., χ by itself is a derivation, and so is θ by itself. We can obtain a new derivation from these by applying, say, the $\wedge$Intro rule,

$$\cfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any sentences, e.g., by χ and θ. Then the conclusion would be $\chi \wedge \theta$, and so

$$\cfrac{\chi \qquad \theta}{\chi \wedge \theta}\ \wedge\text{Intro}$$

is a correct derivation. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ. Thus,

$$\cfrac{\theta \qquad \chi}{\theta \wedge \chi}\ \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, $\rightarrow$Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the conclusion of that conditional. So both of the following would be correct derivations:

$$1\ \cfrac{\cfrac{[\chi]^1 \qquad \theta}{\chi \wedge \theta}\ \wedge\text{Intro}}{\chi \rightarrow (\chi \wedge \theta)}\ \rightarrow\text{Intro} \qquad 1\ \cfrac{\cfrac{\chi \qquad [\theta]^1}{\chi \wedge \theta}\ \wedge\text{Intro}}{\theta \rightarrow (\chi \wedge \theta)}\ \rightarrow\text{Intro}$$

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Bibliography