ntd.1 Derivations

fol:ntd:der:sec We've said what an assumption is, and we've given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference. explanation

Definition ntd.1 (Derivation). A *derivation* of a sentence φ from assumptions Γ is a finite tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in Γ or are discharged by an inference in the tree.
2. The bottommost sentence of the tree is φ.
3. Every sentence in the tree except the sentence φ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that φ is the *conclusion* of the derivation and Γ its undischarged assumptions.

If a derivation of φ from Γ exists, we say that φ is *derivable* from Γ, or in symbols: $\Gamma \vdash \varphi$. If there is a derivation of φ in which every assumption is discharged, we write $\vdash \varphi$.

Example ntd.2. Every assumption on its own is a derivation. So, e.g., φ by itself is a derivation, and so is ψ by itself. We can obtain a new derivation from these by applying, say, the $\land$Intro rule,

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\land\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any sentences, e.g., by χ and θ. Then the conclusion would be $\chi \land \theta$, and so

$$\frac{\chi \qquad \theta}{\chi \land \theta}\land\text{Intro}$$

is a correct derivation. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ. Thus,

$$\frac{\theta \qquad \chi}{\theta \land \chi}\land\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, $\to$Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the antecedent of that conditional. So both of the following would be correct derivations:

$$
\dfrac{\dfrac{[\chi]^1 \qquad \theta}{\chi \land \theta}\,\land\text{Intro}}{\chi \to (\chi \land \theta)}\,{\to}\text{Intro}^{1}
\qquad
\dfrac{\dfrac{\chi \qquad [\theta]^1}{\chi \land \theta}\,\land\text{Intro}}{\theta \to (\chi \land \theta)}\,{\to}\text{Intro}^{1}
$$

They show, respectively, that $\theta \vdash \chi \to (\chi \land \theta)$ and $\chi \vdash \theta \to (\chi \land \theta)$.

Remember that discharging of assumptions is a permission, not a requirement: we don't have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption φ to be discharged:

$$
{}_1\,\dfrac{\psi}{\varphi \to \psi}\,{\to}\text{Intro}
$$

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Bibliography