Derivations

We’ve said what an assumption is, and we’ve given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference.

Definition ntd.1 (Derivation). A derivation of a sentence ϕ from assumptions Γ is a tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in Γ or are discharged by an inference in the tree.

2. The bottommost sentence of the tree is ϕ.

3. Every sentence in the tree except the sentence ϕ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that ϕ is the conclusion of the derivation and that ϕ is derivable from Γ.

Example ntd.2. Every assumption on its own is a derivation. So, e.g., χ by itself is a derivation, and so is θ by itself. We can obtain a new derivation from these by applying, say, the ∧ Intro rule,

\[
\frac{\varphi \quad \psi}{\varphi \land \psi} \land \text{Intro}
\]

These rules are meant to be general: we can replace the ϕ and ψ in it with any sentences, e.g., by χ and θ. Then the conclusion would be χ ∧ θ, and so

\[
\frac{\chi \quad \theta}{\chi \land \theta} \land \text{Intro}
\]

is a correct derivation. Of course, we can also switch the assumptions, so that θ plays the role of ϕ and χ that of ψ. Thus,

\[
\frac{\theta \quad \chi}{\theta \land \chi} \land \text{Intro}
\]

is also a correct derivation.

We can now apply another rule, say, → Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the antecedent of that conditional. So both of the following would be correct derivations:

\[
\frac{[\chi]^1}{\chi \land \theta} \land \text{Intro} \quad \frac{\chi \land \theta}{\chi \rightarrow (\chi \land \theta)} \rightarrow \text{Intro} \quad \frac{[\theta]^1}{\theta \rightarrow (\chi \land \theta)} \rightarrow \text{Intro}
\]
Remember that discharging of assumptions is a permission, not a requirement: we don’t have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption $\varphi$ to be discharged:

$1$ \quad \begin{array}{c} \psi \\ \varphi \rightarrow \psi \end{array} \rightarrow \text{Intro}

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Bibliography