We’ve said what an assumption is, and we’ve given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference.

**Definition ntd.1 (Derivation).** A *derivation* of a sentence $\varphi$ from assumptions $\Gamma$ is a tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in $\Gamma$ or are discharged by an inference in the tree.
2. The bottommost sentence of the tree is $\varphi$.
3. Every sentence in the tree except $\varphi$ is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that $\varphi$ is the conclusion of the derivation and that $\varphi$ is derivable from $\Gamma$.

**Example ntd.2.** Every assumption on its own is a derivation. So, e.g., $\chi$ by itself is a derivation, and so is $\theta$ by itself. We can obtain a new derivation from these by applying, say, the $\land$ Intro rule,

$$
\frac{\varphi \quad \psi}{\varphi \land \psi} \land\text{Intro}
$$

These rules are meant to be general: we can replace the $\varphi$ and $\psi$ in it with any sentences, e.g., by $\chi$ and $\theta$. Then the conclusion would be $\chi \land \theta$, and so

$$
\frac{\chi \quad \theta}{\chi \land \theta} \land\text{Intro}
$$

is a correct derivation. Of course, we can also switch the assumptions, so that $\theta$ plays the role of $\varphi$ and $\chi$ that of $\psi$. Thus,

$$
\frac{\theta \quad \chi}{\theta \land \chi} \land\text{Intro}
$$

is also a correct derivation.

We can now apply another rule, say, $\rightarrow$ Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the conclusion of that conditional. So both of the following would be correct derivations:

$$
\frac{\chi \land \theta}{\chi \rightarrow (\chi \land \theta)} \rightarrow\text{Intro}
\frac{\theta}{\chi \rightarrow (\chi \land \theta)} \rightarrow\text{Intro}
\frac{\chi \land \theta}{\theta \rightarrow (\chi \land \theta)} \rightarrow\text{Intro}
$$