mat.1 Expressing the Size of Structures

fol:mat:siz:sec There are some properties of structures we can express even without using the non-logical symbols of a language. For instance, there are sentences which are true in a structure iff the domain of the structure has at least, at most, or exactly a certain number n of elements. explanation

Proposition mat.1. *The sentence*

$$\begin{aligned}\varphi_{\geq n} \equiv \exists x_1\, \exists x_2\, \dots \exists x_n \quad (&x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \dots \land x_1 \neq x_n \land \\ &x_2 \neq x_3 \land x_2 \neq x_4 \land \dots \land x_2 \neq x_n \land \\ &\vdots \\ &x_{n-1} \neq x_n)\end{aligned}$$

is true in a structure $\mathfrak{M}$ *iff* $|\mathfrak{M}|$ *contains at least* n *elements. Consequently,* $\mathfrak{M} \vDash \lnot\varphi_{\geq n+1}$ *iff* $|\mathfrak{M}|$ *contains at most* n *elements.*

Proposition mat.2. *The sentence*

$$\begin{aligned}\varphi_{= n} \equiv \exists x_1\, \exists x_2\, \dots \exists x_n \quad (&x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \dots \land x_1 \neq x_n \land \\ &x_2 \neq x_3 \land x_2 \neq x_4 \land \dots \land x_2 \neq x_n \land \\ &\vdots \\ &x_{n-1} \neq x_n \land \\ &\forall y\, (y = x_1 \lor \dots y = x_n) \dots))\end{aligned}$$

is true in a structure $\mathfrak{M}$ *iff* $|\mathfrak{M}|$ *contains exactly* n *elements.*

Proposition mat.3. *A structure is infinite iff it is a model of*

$$\{\varphi_{\geq 1}, \varphi_{\geq 2}, \varphi_{\geq 3}, \dots\}$$

There is no single purely logical sentence which is true in $\mathfrak{M}$ iff $|\mathfrak{M}|$ is infinite. However, one can give sentences with non-logical predicate symbols which only have infinite models (although not every infinite structure is a model of them). The property of being a finite structure, and the property of being a non-enumerable structure cannot even be expressed with an infinite set of sentences. These facts follow from the compactness and Löwenheim-Skolem theorems.

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Bibliography