mat.1 Expressing the Size of Structures

There are some properties of structures we can express even without using the non-logical symbols of a language. For instance, there are sentences which are true in a structure iff the domain of the structure has at least, at most, or exactly a certain number $n$ of elements.

Proposition mat.1. The sentence

$$\varphi \geq n \equiv \exists x_1 \exists x_2 \ldots \exists x_n \left( x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \ldots \land x_1 \neq x_n \land \\
 x_2 \neq x_3 \land x_2 \neq x_4 \land \ldots \land x_2 \neq x_n \land \\
 \vdots \\
 x_{n-1} \neq x_n \right)$$

is true in a structure $\mathcal{M}$ iff $|\mathcal{M}|$ contains at least $n$ elements. Consequently, $\mathcal{M} \models \neg \varphi_{\geq n+1}$ iff $|\mathcal{M}|$ contains at most $n$ elements.

Proposition mat.2. The sentence

$$\varphi = n \equiv \exists x_1 \exists x_2 \ldots \exists x_n \left( x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \ldots \land x_1 \neq x_n \land \\
 x_2 \neq x_3 \land x_2 \neq x_4 \land \ldots \land x_2 \neq x_n \land \\
 \vdots \\
 x_{n-1} \neq x_n \land \\
 \forall y \left( y = x_1 \lor \ldots \lor y = x_n \right) \right)$$

is true in a structure $\mathcal{M}$ iff $|\mathcal{M}|$ contains exactly $n$ elements.

Proposition mat.3. A structure is infinite iff it is a model of

$$\{ \varphi_1, \varphi_2, \varphi_3, \ldots \}.$$  

There is no single purely logical sentence which is true in $\mathcal{M}$ iff $|\mathcal{M}|$ is infinite. However, one can give sentences with non-logical predicate symbols which only have infinite models (although not every infinite structure is a model of them). The property of being a finite structure, and the property of being a non-enumerable structure cannot even be expressed with an infinite set of sentences. These facts follow from the compactness and L"owenheim-Skolem theorems.

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