mat.1 Expressing the Size of Structures

There are some properties of structures we can express even without using the non-logical symbols of a language. For instance, there are sentences which are true in a structure iff the domain of the structure has at least, at most, or exactly a certain number \( n \) of elements.

**Proposition mat.1.** The sentence

\[
\varphi_{\geq n} \equiv \exists x_1 \exists x_2 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \cdots \land x_1 \neq x_n \land x_2 \neq x_3 \land x_2 \neq x_4 \land \cdots \land x_2 \neq x_n \land \cdots x_{n-1} \neq x_n)
\]

is true in a structure \( M \) iff \( |M| \) contains at least \( n \) elements. Consequently, \( M \models \neg \varphi_{\geq n+1} \) iff \( |M| \) contains at most \( n \) elements.

**Proposition mat.2.** The sentence

\[
\varphi_{= n} \equiv \exists x_1 \exists x_2 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land x_1 \neq x_4 \land \cdots \land x_1 \neq x_n \land x_2 \neq x_3 \land x_2 \neq x_4 \land \cdots \land x_2 \neq x_n \land \cdots x_{n-1} \neq x_n \land \forall y (y = x_1 \lor \cdots \lor y = x_n))
\]

is true in a structure \( M \) iff \( |M| \) contains exactly \( n \) elements.

**Proposition mat.3.** A structure is infinite iff it is a model of

\[
\{ \varphi_{\geq 1}, \varphi_{\geq 2}, \varphi_{\geq 3}, \ldots \}.
\]

There is no single purely logical sentence which is true in \( M \) iff \( |M| \) is infinite. However, one can give sentences with non-logical predicate symbols which only have infinite models (although not every infinite structure is a model of them). The property of being a finite structure, and the property of being a non-enumerable structure cannot even be expressed with an infinite set of sentences. These facts follow from the compactness and Löwenheim–Skolem theorems.
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Bibliography