

## mat.1 Expressing the Size of Structures

fol:mat:siz: There are some properties of structures we can express even without using the explanation  
 sec non-logical symbols of a language. For instance, there are sentences which are true in a structure iff the domain of the structure has at least, at most, or exactly a certain number  $n$  of elements.

**Proposition mat.1.** *The sentence*

$$\begin{aligned} \varphi_{\geq n} \equiv & \exists x_1 \exists x_2 \dots \exists x_n \\ & (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge \dots \wedge x_1 \neq x_n \wedge \\ & \quad x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge \dots \wedge x_2 \neq x_n \wedge \\ & \quad \vdots \\ & \quad x_{n-1} \neq x_n) \end{aligned}$$

is true in a structure  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  contains at least  $n$  elements. Consequently,  $\mathfrak{M} \models \neg\varphi_{\geq n+1}$  iff  $|\mathfrak{M}|$  contains at most  $n$  elements.

**Proposition mat.2.** *The sentence*

$$\begin{aligned} \varphi_{=n} \equiv & \exists x_1 \exists x_2 \dots \exists x_n \\ & (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge \dots \wedge x_1 \neq x_n \wedge \\ & \quad x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge \dots \wedge x_2 \neq x_n \wedge \\ & \quad \vdots \\ & \quad x_{n-1} \neq x_n \wedge \\ & \quad \forall y (y = x_1 \vee \dots \vee y = x_n)) \end{aligned}$$

is true in a structure  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  contains exactly  $n$  elements.

**Proposition mat.3.** *A structure is infinite iff it is a model of*

$$\{\varphi_{\geq 1}, \varphi_{\geq 2}, \varphi_{\geq 3}, \dots\}.$$

There is no single purely logical sentence which is true in  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is infinite. However, one can give sentences with non-logical predicate symbols which only have infinite models (although not every infinite structure is a model of them). The property of being a finite structure, and the property of being a non-enumerable structure cannot even be expressed with an infinite set of sentences. These facts follow from the compactness and Löwenheim–Skolem theorems.

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**Bibliography**