mat.1 Expressing Relations in a Structure

fol:mat:exr:sec One main use formulas can be put to is to express properties and relations in a structure $\mathfrak{M}$ in terms of the primitives of the language $\mathcal{L}$ of $\mathfrak{M}$. By this we mean the following: the domain of $\mathfrak{M}$ is a set of objects. The constant symbols, function symbols, and predicate symbols are interpreted in $\mathfrak{M}$ by some objects in$|\mathfrak{M}|$, functions on $|\mathfrak{M}|$, and relations on $|\mathfrak{M}|$. For instance, if A^2_0 is in $\mathcal{L}$, then $\mathfrak{M}$ assigns to it a relation $R = A^{2\,\mathfrak{M}}_0$. Then the formula $A^2_0(v_1, v_2)$ *expresses* that very relation, in the following sense: if a variable assignment s maps v_1 to $a \in |\mathfrak{M}|$ and v_2 to $b \in |\mathfrak{M}|$, then explanation

$$Rab \quad \text{iff} \quad \mathfrak{M}, s \vDash A^2_0(v_1, v_2).$$

Note that we have to involve variable assignments here: we can't just say "Rab iff $\mathfrak{M} \vDash A^2_0(a, b)$" because a and b are not symbols of our language: they are elements of $|\mathfrak{M}|$.

Since we don't just have atomic formulas, but can combine them using the logical connectives and the quantifiers, more complex formulas can define other relations which aren't directly built into $\mathfrak{M}$. We're interested in how to do that, and specifically, which relations we can define in a structure.

Definition mat.1. Let $\varphi(v_1, \dots, v_n)$ be a formula of $\mathcal{L}$ in which only v_1,..., v_n occur free, and let $\mathfrak{M}$ be a structure for $\mathcal{L}$. $\varphi(v_1, \dots, v_n)$ *expresses the relation* $R \subseteq |\mathfrak{M}|^n$ iff

$$Ra_1 \dots a_n \quad \text{iff} \quad \mathfrak{M}, s \vDash \varphi(v_1, \dots, v_n)$$

for any variable assignment s with $s(v_i) = a_i$ $(i = 1, \dots, n)$.

Example mat.2. In the standard model of arithmetic $\mathfrak{N}$, the formula $v_1 < v_2 \lor v_1 = v_2$ expresses the $\leq$ relation on $\mathbb{N}$. The formula $v_2 = v_1'$ expresses the successor relation, i.e., the relation $R \subseteq \mathbb{N}^2$ where Rnm holds if m is the successor of n. The formula $v_1 = v_2'$ expresses the predecessor relation. The formulas $\exists v_3\,(v_3 \neq 0 \land v_2 = (v_1 + v_3))$ and $\exists v_3\,(v_1 + v_3{}') = v_2$ both express the $<$ relation. This means that the predicate symbol $<$ is actually superfluous in the language of arithmetic; it can be defined.

This idea is not just interesting in specific structures, but generally whenever we use a language to describe an intended model or models, i.e., when we consider theories. These theories often only contain a few predicate symbols as basic symbols, but in the domain they are used to describe often many other relations play an important role. If these other relations can be systematically expressed by the relations that interpret the basic predicate symbols of the language, we say we can *define* them in the language. explanation

Problem mat.1. Find formulas in $\mathcal{L}_A$ which define the following relations:

1. n is between i and j;

2. n evenly divides m (i.e., m is a multiple of n);

3. n is a prime number (i.e., no number other than 1 and n evenly divides n).

Problem mat.2. Suppose the formula $\varphi(v_1, v_2)$ expresses the relation $R \subseteq |\mathfrak{M}|^2$ in a structure $\mathfrak{M}$. Find formulas that express the following relations:

1. the inverse R^{-1} of R;

2. the relative product $R \mid R$;

Can you find a way to express R^+, the transitive closure of R?

Problem mat.3. Let $\mathcal{L}$ be the language containing a 2-place predicate symbol $<$ only (no other constant symbols, function symbols or predicate symbols—except of course $=$). Let $\mathfrak{N}$ be the structure such that $|\mathfrak{N}| = \mathbb{N}$, and $<^{\mathfrak{N}} = \{\langle n, m \rangle : n < m\}$. Prove the following:

1. $\{0\}$ is definable in $\mathfrak{N}$;

2. $\{1\}$ is definable in $\mathfrak{N}$;

3. $\{2\}$ is definable in $\mathfrak{N}$;

4. for each $n \in \mathbb{N}$, the set $\{n\}$ is definable in $\mathfrak{N}$;

5. every finite subset of $|\mathfrak{N}|$ is definable in $\mathfrak{N}$;

6. every co-finite subset of $|\mathfrak{N}|$ is definable in $\mathfrak{N}$ (where $X \subseteq \mathbb{N}$ is co-finite iff $\mathbb{N} \setminus X$ is finite).

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Bibliography