

mat.1 Expressing Properties of Structures

fol:mat:exs:
sec

It is often useful and important to express conditions on functions and relations, or more generally, that the functions and relations in a structure satisfy these conditions. For instance, we would like to have ways of distinguishing those structures for a language which “capture” what we want the predicate symbols to “mean” from those that do not. Of course we’re completely free to specify which structures we “intend,” e.g., we can specify that the interpretation of the predicate symbol \leq must be an ordering, or that we are only interested in interpretations of \mathcal{L} in which the domain consists of sets and \in is interpreted by the “is an element of” relation. But can we do this with sentences of the language? In other words, which conditions on a structure \mathfrak{M} can we express by a sentence (or perhaps a set of sentences) in the language of \mathfrak{M} ? There are some conditions that we will not be able to express. For instance, there is no sentence of \mathcal{L}_A which is only true in a structure \mathfrak{M} if $|\mathfrak{M}| = \mathbb{N}$. We cannot express “the domain contains only natural numbers.” But there are “structural properties” of structures that we perhaps can express. Which properties of structures can we express by sentences? Or, to put it another way, which collections of structures can we describe as those making a sentence (or set of sentences) true?

explanation

Definition mat.1 (Model of a set). Let Γ be a set of sentences in a language \mathcal{L} . We say that a structure \mathfrak{M} is a model of Γ if $\mathfrak{M} \models \varphi$ for all $\varphi \in \Gamma$.

Example mat.2. The sentence $\forall x x \leq x$ is true in \mathfrak{M} iff $\leq^{\mathfrak{M}}$ is a reflexive relation. The sentence $\forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y)$ is true in \mathfrak{M} iff $\leq^{\mathfrak{M}}$ is anti-symmetric. The sentence $\forall x \forall y \forall z ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$ is true in \mathfrak{M} iff $\leq^{\mathfrak{M}}$ is transitive. Thus, the models of

$$\left\{ \begin{array}{l} \forall x x \leq x, \\ \forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y), \\ \forall x \forall y \forall z ((x \leq y \wedge y \leq z) \rightarrow x \leq z) \end{array} \right\}$$

are exactly those structures in which $\leq^{\mathfrak{M}}$ is reflexive, anti-symmetric, and transitive, i.e., a partial order. Hence, we can take them as axioms for the first-order theory of partial orders.

Photo Credits

Bibliography