int.1 Substitution

We’ll discuss an example to illustrate how things hang together, and how the development of syntax and semantics lays the foundation for our more advanced investigations later. Our derivation systems should let us derive $P(a)$ from $\forall v_0 P(v_0)$. Maybe we even want to state this as a rule of inference. However, to do so, we must be able to state it in the most general terms: not just for $P$, $a$, and $v_0$, but for any formula $\varphi$, and term $t$, and variable $x$. (Recall that constant symbols are terms, but we’ll consider also more complicated terms built from constant symbols and function symbols.) So we want to be able to say something like, “whenever you have derived $\forall x \varphi(x)$ you are justified in inferring $\varphi(t)$—the result of removing $\forall x$ and replacing $x$ by $t$.” But what exactly does “replacing $x$ by $t$” mean? What is the relation between $\varphi(x)$ and $\varphi(t)$? Does this always work?

To make this precise, we define the operation of substitution. Substitution is actually tricky, because we can’t just replace all $x$’s in $\varphi$ by $t$, and not every $t$ can be substituted for any $x$. We’ll deal with this, again, using inductive definitions. But once this is done, specifying an inference rule as “infer $\varphi(t)$ from $\forall x \varphi(x)$” becomes a precise definition. Moreover, we’ll be able to show that this is a good inference rule in the sense that $\forall x \varphi(x)$ entails $\varphi(t)$. But to prove this, we have to again prove something that may at first glance prompt you to ask “why are we doing this?” That $\forall x \varphi(x)$ entails $\varphi(t)$ relies on the fact that whether or not $\mathcal{M} \models \varphi(t)$ holds depends only on the value of the term $t$, i.e., if we let $m$ be whatever element of $|\mathcal{M}|$ is picked out by $t$, then $\mathcal{M}, s \models \varphi(t)$ iff $\mathcal{M}, s[m/x] \models \varphi(x)$. This holds even when $t$ contains variables, but we’ll have to be careful with how exactly we state the result.

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Bibliography