

int.1 Substitution

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We'll discuss an example to illustrate how things hang together, and how the development of syntax and semantics lays the foundation for our more advanced investigations later. Our **derivation** systems should let us **derive** $P(a)$ from $\forall v_0 P(v_0)$. Maybe we even want to state this as a rule of inference. However, to do so, we must be able to state it in the most general terms: not just for P , a , and v_0 , but for any **formula** φ , and term t , and **variable** x . (Recall that **constant symbols** are terms, but we'll consider also more complicated terms built from **constant symbols** and **function symbols**.) So we want to be able to say something like, "whenever you have **derived** $\forall x \varphi(x)$ you are justified in inferring $\varphi(t)$ —the result of removing $\forall x$ and replacing x by t ." But what exactly does "replacing x by t " mean? What is the relation between $\varphi(x)$ and $\varphi(t)$? Does this always work?

To make this precise, we define the operation of *substitution*. Substitution is actually tricky, because we can't just replace all x 's in φ by t , and not every t can be substituted for any x . We'll deal with this, again, using inductive definitions. But once this is done, specifying an inference rule as "infer $\varphi(t)$ from $\forall x \varphi(x)$ " becomes a precise definition. Moreover, we'll be able to show that this is a good inference rule in the sense that $\forall x \varphi(x)$ entails $\varphi(t)$. But to prove this, we have to again prove something that may at first glance prompt you to ask "why are we doing this?" That $\forall x \varphi(x)$ entails $\varphi(t)$ relies on the fact that whether or not $\mathfrak{M} \models \varphi(t)$ holds depends only on the value of the term t , i.e., if we let m be whatever **element** of $|\mathfrak{M}|$ is picked out by t , then $\mathfrak{M}, s \models \varphi(t)$ iff $\mathfrak{M}, s[m/x] \models \varphi(x)$. This holds even when t contains **variables**, but we'll have to be careful with how exactly we state the result.

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Bibliography