

int.1 Models and Theories

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Once we've defined the syntax and semantics of first-order logic, we can get to work investigating the properties of **structures** and the semantic notions. We can also define **derivation** systems, and investigate those. For a set of **sentences**, we can ask: what **structures** make all the **sentences** in that set true? Given a set of **sentences** Γ , a **structure** \mathfrak{M} that satisfies them is called a *model of Γ* . We might start from Γ and try to find its models—what do they look like? How big or small do they have to be? But we might also start with a single **structure** or collection of **structures** and ask: what **sentences** are true in them? Are there **sentences** that *characterize* these **structures** in the sense that they, and only they, are true in them? These kinds of questions are the domain of *model theory*. They also underlie the *axiomatic method*: describing a collection of **structures** by a set of **sentences**, the axioms of a theory. This is made possible by the observation that exactly those **sentences** entailed in first-order logic by the axioms are true in all models of the axioms.

As a very simple example, consider preorders. A preorder is a relation R on some set A which is both reflexive and transitive. A set A with a two-place relation $R \subseteq A \times A$ on it is exactly what we would need to give a **structure** for a first-order language with a single two-place relation symbol P : we would set $|\mathfrak{M}| = A$ and $P^{\mathfrak{M}} = R$. Since R is a preorder, it is reflexive and transitive, and we can find a set Γ of **sentences** of first-order logic that say this:

$$\begin{aligned} &\forall v_0 P(v_0, v_0) \\ &\forall v_0 \forall v_1 \forall v_2 ((P(v_0, v_1) \wedge P(v_1, v_2)) \rightarrow P(v_0, v_2)) \end{aligned}$$

These **sentences** are just the symbolizations of “for any x , Rxx ” (R is reflexive) and “whenever Rxy and Ryz then also Rxz ” (R is transitive). We see that a **structure** \mathfrak{M} is a model of these two **sentences** Γ iff R (i.e., $P^{\mathfrak{M}}$), is a preorder on A (i.e., $|\mathfrak{M}|$). In other words, the models of Γ are exactly the preorders. Any property of all preorders that can be expressed in the first-order language with just P as **predicate symbol** (like reflexivity and transitivity above), is entailed by the two **sentences** in Γ and vice versa. So anything we can prove about models of Γ we have proved about all preorders.

For any particular theory and class of models (such as Γ and all preorders), there will be interesting questions about what can be expressed in the corresponding first-order language, and what cannot be expressed. There are some properties of **structures** that are interesting for all languages and classes of models, namely those concerning the size of the **domain**. One can always express, for instance, that the **domain** contains exactly n **elements**, for any $n \in \mathbb{Z}^+$. One can also express, using a set of infinitely many **sentences**, that the **domain** is infinite. But one cannot express that the domain is finite, or that the domain is **non-enumerable**. These results about the limitations of first-order languages are consequences of the compactness and Löwenheim–Skolem theorems.

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Bibliography