Lemma com.1 (Lindenbaum’s Lemma). Every consistent set $\Gamma$ in a language $L$ can be extended to a complete and consistent set $\Gamma^*$.

Proof. Let $\Gamma$ be consistent. Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all the sentences of $L$. Define $\Gamma_0 = \Gamma$, and

$$
\Gamma_{n+1} = \begin{cases} 
\Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\
\Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.}
\end{cases}
$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$.

Each $\Gamma_n$ is consistent: $\Gamma_0$ is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, this is because the latter is consistent. If it isn’t, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$. We have to verify that $\Gamma_n \cup \{\neg\varphi_n\}$ is consistent. Suppose it’s not. Then both $\Gamma_n \cup \{\varphi_n\}$ and $\Gamma_n \cup \{\neg\varphi_n\}$ are inconsistent. This means that $\Gamma_n$ would be inconsistent by ??????????????, contrary to the induction hypothesis.

For every $n$ and every $i < n$, $\Gamma_i \subseteq \Gamma_n$. This follows by a simple induction on $n$. For $n = 0$, there are no $i < 0$, so the claim holds automatically. For the inductive step, suppose it is true for $n$. We have $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ or $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$ by construction. So $\Gamma_n \subseteq \Gamma_{n+1}$. If $i < n$, then $\Gamma_i \subseteq \Gamma_n$ by inductive hypothesis, and so $\Gamma_i \subseteq \Gamma_{n+1}$ by transitivity of $\subseteq$.

From this it follows that every finite subset of $\Gamma^*$ is a subset of $\Gamma_n$ for some $n$, since each $\psi \in \Gamma^*$ not already in $\Gamma_0$ is added at some stage $i$. If $n$ is the last one of these, then all $\psi$ in the finite subset are in $\Gamma_n$. So, every finite subset of $\Gamma^*$ is consistent. By ??????????????, $\Gamma^*$ is consistent.

Every sentence of $\Frm(L)$ appears on the list used to define $\Gamma^*$. If $\varphi_n \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi_n\}$ was inconsistent. But then $\neg\varphi_n \in \Gamma^*$, so $\Gamma^*$ is complete. \hfill $\Box$

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Bibliography