

com.1 Lindenbaum's Lemma

fol:com:lin:
sec We now prove a lemma that shows that any consistent set of **sentences** explanation is contained in some set of sentences which is not just consistent, but also **complete**. The proof works by adding one **sentence** at a time, guaranteeing at each step that the set remains consistent. We do this so that for every φ , either φ or $\neg\varphi$ gets added at some stage. The union of all stages in that construction then contains either φ or its negation $\neg\varphi$ and is thus complete. It is also consistent, since we made sure at each stage not to introduce an inconsistency.

fol:com:lin:
lem:lindenbaum **Lemma com.1** (Lindenbaum's Lemma). *Every consistent set Γ in a language \mathcal{L} can be extended to a **complete** and consistent set Γ^* .*

Proof. Let Γ be consistent. Let $\varphi_0, \varphi_1, \dots$ be an enumeration of all the **sentences** of \mathcal{L} . Define $\Gamma_0 = \Gamma$, and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$.

Each Γ_n is consistent: Γ_0 is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, this is because the latter is consistent. If it isn't, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$. We have to verify that $\Gamma_n \cup \{\neg\varphi_n\}$ is consistent. Suppose it's not. Then *both* $\Gamma_n \cup \{\varphi_n\}$ and $\Gamma_n \cup \{\neg\varphi_n\}$ are inconsistent. This means that Γ_n would be inconsistent by **????????????????**, contrary to the induction hypothesis.

For every n and every $i < n$, $\Gamma_i \subseteq \Gamma_n$. This follows by a simple induction on n . For $n = 0$, there are no $i < 0$, so the claim holds automatically. For the inductive step, suppose it is true for n . We have $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ or $= \Gamma_n \cup \{\neg\varphi_n\}$ by construction. So $\Gamma_n \subseteq \Gamma_{n+1}$. If $i < n$, then $\Gamma_i \subseteq \Gamma_n$ by inductive hypothesis, and so $\subseteq \Gamma_{n+1}$ by transitivity of \subseteq .

From this it follows that every finite subset of Γ^* is a subset of Γ_n for some n , since each $\psi \in \Gamma^*$ not already in Γ_0 is added at some stage i . If n is the last one of these, then all ψ in the finite subset are in Γ_n . So, every finite subset of Γ^* is consistent. By **????????????????**, Γ^* is consistent.

Every **sentence** of $\text{Frm}(\mathcal{L})$ appears on the list used to define Γ^* . If $\varphi_n \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi_n\}$ was inconsistent. But then $\neg\varphi_n \in \Gamma^*$, so Γ^* is **complete**. \square

Photo Credits

Bibliography