

com.1 Introduction

fol:com:int:
sec

The completeness theorem is one of the most fundamental results about logic. It comes in two formulations, the equivalence of which we'll prove. In its first formulation it says something fundamental about the relationship between semantic consequence and our proof system: if a **sentence** φ follows from some **sentences** Γ , then there is also a **derivation** that establishes $\Gamma \vdash \varphi$. Thus, the proof system is as strong as it can possibly be without proving things that don't actually follow.

In its second formulation, it can be stated as a model existence result: every consistent set of **sentences** is satisfiable. Consistency is a proof-theoretic notion: it says that our proof system is unable to produce certain **derivations**. But who's to say that just because there are no **derivations** of a certain sort from Γ , it's guaranteed that there is a **structure** \mathfrak{M} with $\mathfrak{M} \models \Gamma$? Before the completeness theorem was first proved—in fact before we had the proof systems we now do—the great German mathematician David Hilbert held the view that consistency of mathematical theories guarantees the existence of the objects they are about. He put it as follows in a letter to Gottlob Frege:

If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Frege vehemently disagreed. The second formulation of the completeness theorem shows that Hilbert was right in at least the sense that if the axioms are consistent, then *some* **structure** exists that makes them all true.

These aren't the only reasons the completeness theorem—or rather, its proof—is important. It has a number of important consequences, some of which we'll discuss separately. For instance, since any **derivation** that shows $\Gamma \vdash \varphi$ is finite and so can only use finitely many of the **sentences** in Γ , it follows by the completeness theorem that if φ is a consequence of Γ , it is already a consequence of a finite subset of Γ . This is called *compactness*. Equivalently, if every finite subset of Γ is consistent, then Γ itself must be consistent.

Although the compactness theorem follows from the completeness theorem via the detour through **derivations**, it is also possible to use the *the proof of* the completeness theorem to establish it directly. For what the proof does is take a set of **sentences** with a certain property—consistency—and constructs a **structure** out of this set that has certain properties (in this case, that it satisfies the set). Almost the very same construction can be used to directly establish compactness, by starting from “finitely satisfiable” sets of **sentences** instead of consistent ones. The construction also yields other consequences, e.g., that any satisfiable set of **sentences** has a finite or **denumerable** model. (This result is called the Löwenheim-Skolem theorem.) In general, the construction of **structures** from sets of **sentences** is used often in logic, and sometimes even in philosophy.

Photo Credits

Bibliography