

com.1 Identity

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sec

The construction of the term model given in the preceding section is enough explanation to establish completeness for first-order logic for sets Γ that do not contain $=$. The term model satisfies every $\varphi \in \Gamma^*$ which does not contain $=$ (and hence all $\varphi \in \Gamma$). It does not work, however, if $=$ is present. The reason is that Γ^* then may contain a sentence $t = t'$, but in the term model the value of any term is that term itself. Hence, if t and t' are different terms, their values in the term model—i.e., t and t' , respectively—are different, and so $t = t'$ is false. We can fix this, however, using a construction known as “factoring.”

Definition com.1. Let Γ^* be a consistent and complete set of sentences in \mathcal{L} . We define the relation \approx on the set of closed terms of \mathcal{L} by

$$t \approx t' \quad \text{iff} \quad t = t' \in \Gamma^*$$

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prop:approx-equiv

Proposition com.2. *The relation \approx has the following properties:*

1. \approx is reflexive.
2. \approx is symmetric.
3. \approx is transitive.
4. If $t \approx t'$, f is a function symbol, and $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ are terms, then

$$f(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \approx f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n).$$

5. If $t \approx t'$, R is a predicate symbol, and $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ are terms, then

$$R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \in \Gamma^* \quad \text{iff} \\ R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n) \in \Gamma^*.$$

Proof. Since Γ^* is consistent and complete, $t = t' \in \Gamma^*$ iff $\Gamma^* \vdash t = t'$. Thus it is enough to show the following:

1. $\Gamma^* \vdash t = t$ for all terms t .
2. If $\Gamma^* \vdash t = t'$ then $\Gamma^* \vdash t' = t$.
3. If $\Gamma^* \vdash t = t'$ and $\Gamma^* \vdash t' = t''$, then $\Gamma^* \vdash t = t''$.
4. If $\Gamma^* \vdash t = t'$, then

$$\Gamma^* \vdash f(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) = f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$$

for every n -place function symbol f and terms $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$.

5. If $\Gamma^* \vdash t = t'$ and $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n)$, then $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$ for every n -place **predicate symbol** R and terms $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$.

□

Problem com.1. Complete the proof of **Proposition com.2**.

Definition com.3. Suppose Γ^* is a consistent and **complete** set in a language \mathcal{L} , t is a term, and \approx as in the previous definition. Then:

$$[t]_{\approx} = \{t' : t' \in \text{Trm}(\mathcal{L}), t \approx t'\}$$

and $\text{Trm}(\mathcal{L})/\approx = \{[t]_{\approx} : t \in \text{Trm}(\mathcal{L})\}$.

Definition com.4. Let $\mathfrak{M} = \mathfrak{M}(\Gamma^*)$ be the term model for Γ^* . Then \mathfrak{M}/\approx is the following **structure**:

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defn:term-model-factor

1. $|\mathfrak{M}/\approx| = \text{Trm}(\mathcal{L})/\approx$.
2. $c^{\mathfrak{M}/\approx} = [c]_{\approx}$
3. $f^{\mathfrak{M}/\approx}([t_1]_{\approx}, \dots, [t_n]_{\approx}) = [f(t_1, \dots, t_n)]_{\approx}$
4. $\langle [t_1]_{\approx}, \dots, [t_n]_{\approx} \rangle \in R^{\mathfrak{M}/\approx}$ iff $\mathfrak{M} \models R(t_1, \dots, t_n)$.

explanation

Note that we have defined $f^{\mathfrak{M}/\approx}$ and $R^{\mathfrak{M}/\approx}$ for elements of $\text{Trm}(\mathcal{L})/\approx$ by referring to them as $[t]_{\approx}$, i.e., via *representatives* $t \in [t]_{\approx}$. We have to make sure that these definitions do not depend on the choice of these representatives, i.e., that for some other choices t' which determine the same equivalence classes ($[t]_{\approx} = [t']_{\approx}$), the definitions yield the same result. For instance, if R is a one-place **predicate symbol**, the last clause of the definition says that $[t]_{\approx} \in R^{\mathfrak{M}/\approx}$ iff $\mathfrak{M} \models R(t)$. If for some other term t' with $t \approx t'$, $\mathfrak{M} \not\models R(t')$, then the definition would require $[t']_{\approx} \notin R^{\mathfrak{M}/\approx}$. If $t \approx t'$, then $[t]_{\approx} = [t']_{\approx}$, but we can't have both $[t]_{\approx} \in R^{\mathfrak{M}/\approx}$ and $[t]_{\approx} \notin R^{\mathfrak{M}/\approx}$. However, **Proposition com.2** guarantees that this cannot happen.

Proposition com.5. \mathfrak{M}/\approx is well defined, i.e., if $t_1, \dots, t_n, t'_1, \dots, t'_n$ are terms, and $t_i \approx t'_i$ then

1. $[f(t_1, \dots, t_n)]_{\approx} = [f(t'_1, \dots, t'_n)]_{\approx}$, i.e.,

$$f(t_1, \dots, t_n) \approx f(t'_1, \dots, t'_n)$$

and

2. $\mathfrak{M} \models R(t_1, \dots, t_n)$ iff $\mathfrak{M} \models R(t'_1, \dots, t'_n)$, i.e.,

$$R(t_1, \dots, t_n) \in \Gamma^* \text{ iff } R(t'_1, \dots, t'_n) \in \Gamma^*.$$

Proof. Follows from **Proposition com.2** by induction on n . □

fol.com:ide: **Lemma com.6.** $\mathfrak{M}/\approx \models \varphi$ iff $\varphi \in \Gamma^*$ for all sentences φ .
lem:truth

Proof. By induction on φ , just as in the proof of ???. The only case that needs additional attention is when $\varphi \equiv t = t'$.

$$\begin{aligned}\mathfrak{M}/\approx \models t = t' &\text{ iff } [t]_{\approx} = [t']_{\approx} \text{ (by definition of } \mathfrak{M}/\approx) \\ &\text{ iff } t \approx t' \text{ (by definition of } [t]_{\approx}) \\ &\text{ iff } t = t' \in \Gamma^* \text{ (by definition of } \approx).\end{aligned}$$

□

Note that while $\mathfrak{M}(\Gamma^*)$ is always **enumerable** and infinite, \mathfrak{M}/\approx may be finite, since it may turn out that there are only finitely many classes $[t]_{\approx}$. This is to be expected, since Γ may contain **sentences** which require any **structure** in which they are true to be finite. For instance, $\forall x \forall y x = y$ is a consistent **sentence**, but is satisfied only in **structures** with a **domain** that contains exactly one **element**. digression

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Bibliography