

**com.1 Identity**

The construction of the term model given in the preceding section is enough to establish completeness for first-order logic for sets \( \Gamma \) that do not contain \( = \). The term model satisfies every \( \varphi \in \Gamma^* \) which does not contain \( = \) (and hence all \( \varphi \in \Gamma \)). It does not work, however, if \( = \) is present. The reason is that \( \Gamma^* \) then may contain a sentence \( t = t' \), but in the term model the value of any term is that term itself. Hence, if \( t \) and \( t' \) are different terms, their values in the term model—i.e., \( t \) and \( t' \), respectively—are different, and so \( t = t' \) is false. We can fix this, however, using a construction known as “factoring.”

**Definition com.1.** Let \( \Gamma^* \) be a consistent and complete set of sentences in \( \mathcal{L} \). We define the relation \( \approx \) on the set of closed terms of \( \mathcal{L} \) by

\[
t \approx t' \quad \text{iff} \quad t = t' \in \Gamma^*
\]

**Proposition com.2.** The relation \( \approx \) has the following properties:

1. \( \approx \) is reflexive.
2. \( \approx \) is symmetric.
3. \( \approx \) is transitive.
4. If \( t \approx t' \), \( f \) is a function symbol, and \( t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \) are terms, then

\[
f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n) \approx f(t_1, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_n).
\]

5. If \( t \approx t' \), \( R \) is a predicate symbol, and \( t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \) are terms, then

\[
R(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n) \in \Gamma^* \quad \text{iff} \quad R(t_1, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_n) \in \Gamma^*.
\]

**Proof.** Since \( \Gamma^* \) is consistent and complete, \( t = t' \in \Gamma^* \) iff \( \Gamma^* \models t = t' \). Thus it is enough to show the following:

1. \( \Gamma^* \models t = t \) for all terms \( t \).
2. If \( \Gamma^* \models t = t' \) then \( \Gamma^* \models t' = t \).
3. If \( \Gamma^* \models t = t' \) and \( \Gamma^* \models t' = t'' \), then \( \Gamma^* \models t = t'' \).
4. If \( \Gamma^* \models t = t' \), then

\[
\Gamma^* \models f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n) = f(t_1, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_n)
\]

for every \( n \)-place function symbol \( f \) and terms \( t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \).
5. If $\Gamma^* \vdash t = t'$ and $\Gamma^* \vdash R(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$, then $\Gamma^* \vdash R(t_1, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_n)$ for every $n$-place predicate symbol $R$ and terms $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$.

\[\square\]

**Problem com.1.** Complete the proof of Proposition com.2.

**Definition com.3.** Suppose $\Gamma^*$ is a consistent and complete set in a language $L$, $t$ is a term, and $\approx$ as in the previous definition. Then:

$$[t]_\approx = \{t' : t' \in \text{Trm}(L), t \approx t'\}$$

and $\text{Trm}(L)/\approx = \{[t]_\approx : t \in \text{Trm}(L)\}$.

**Definition com.4.** Let $\mathfrak{M} = \mathfrak{M}(\Gamma^*)$ be the term model for $\Gamma^*$. Then $\mathfrak{M}/\approx$ is the following structure:

1. $|\mathfrak{M}/\approx| = \text{Trm}(L)/\approx$.
2. $c^{\mathfrak{M}/\approx}_\approx = [c]_\approx$
3. $f^{\mathfrak{M}/\approx}(\langle [t_1]_\approx, \ldots, [t_n]_\approx \rangle) = [f(t_1, \ldots, t_n)]_\approx$
4. $\langle [t_1]_\approx, \ldots, [t_n]_\approx \rangle \in R^{\mathfrak{M}/\approx}_\approx$ iff $\mathfrak{M} \models R(t_1, \ldots, t_n)$.

Note that we have defined $f^{\mathfrak{M}/\approx}$ and $R^{\mathfrak{M}/\approx}_\approx$ for elements of $\text{Trm}(L)/\approx$ by referring to them as $[t]_\approx$, i.e., via representatives $t \in [t]_\approx$. We have to make sure that these definitions do not depend on the choice of these representatives, i.e., that for some other choices $t'$ which determine the same equivalence classes $([t]_\approx = [t']_\approx)$, the definitions yield the same result. For instance, if $R$ is a one-place predicate symbol, the last clause of the definition says that $[t]_\approx \in R^{\mathfrak{M}/\approx}_\approx$ iff $\mathfrak{M} \models R(t)$. If for some other term $t'$ with $t \approx t'$, $\mathfrak{M} \not\models R(t)$, then the definition would require $[t']_\approx \notin R^{\mathfrak{M}/\approx}_\approx$. If $t \approx t'$, then $[t]_\approx = [t']_\approx$, but we can't have both $[t]_\approx \in R^{\mathfrak{M}/\approx}_\approx$ and $[t]_\approx \notin R^{\mathfrak{M}/\approx}_\approx$. However, Proposition com.2 guarantees that this cannot happen.

**Proposition com.5.** $\mathfrak{M}/\approx$ is well defined, i.e., if $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ are terms, and $t_i \approx t'_i$ then

1. $[f(t_1, \ldots, t_n)]_\approx = [f(t'_1, \ldots, t'_n)]_\approx$, i.e.,

$$f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)$$

and

2. $\mathfrak{M} \models R(t_1, \ldots, t_n)$ iff $\mathfrak{M} \models R(t'_1, \ldots, t'_n)$, i.e.,

$$R(t_1, \ldots, t_n) \in \Gamma^* \iff R(t'_1, \ldots, t'_n) \in \Gamma^*.$$

**Proof.** Follows from Proposition com.2 by induction on $n$. \[\square\]
Lemma com.6. \( \mathcal{M}/\approx \models \varphi \) iff \( \varphi \in \Gamma^* \) for all sentences \( \varphi \).

Proof. By induction on \( \varphi \), just as in the proof of ??\( . \) The only case that needs additional attention is when \( \varphi \equiv t = t' \).

\[ \mathcal{M}/\approx \models t = t' \iff [t]_\approx = [t']_\approx \quad \text{(by definition of } \mathcal{M}/\approx) \]
\[ \iff t \approx t' \quad \text{(by definition of } [t]_\approx) \]
\[ \iff t = t' \in \Gamma^* \quad \text{(by definition of } \approx). \]

Note that while \( \mathfrak{M}(\Gamma^*) \) is always enumerable and infinite, \( \mathcal{M}/\approx \) may be finite, since it may turn out that there are only finitely many classes \([t]_\approx\). This is to be expected, since \( \Gamma \) may contain sentences which require any structure in which they are true to be finite. For instance, \( \forall x \forall y x = y \) is a consistent sentence, but is satisfied only in structures with a domain that contains exactly one element.

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Bibliography