

## com.1 Henkin Expansion

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sec

Part of the challenge in proving the completeness theorem is that the model we construct from a complete consistent set  $\Gamma$  must make all the quantified formulas in  $\Gamma$  true. In order to guarantee this, we use a trick due to Leon Henkin. In essence, the trick consists in expanding the language by infinitely many constant symbols and adding, for each formula with one free variable  $\varphi(x)$  a formula of the form  $\exists x \varphi \rightarrow \varphi(c)$ , where  $c$  is one of the new constant symbols. When we construct the structure satisfying  $\Gamma$ , this will guarantee that each true existential sentence has a witness among the new constants.

explanation

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prop:lang-exp

**Proposition com.1.** *If  $\Gamma$  is consistent in  $\mathcal{L}$  and  $\mathcal{L}'$  is obtained from  $\mathcal{L}$  by adding a denumerable set of new constant symbols  $d_0, d_1, \dots$ , then  $\Gamma$  is consistent in  $\mathcal{L}'$ .*

**Definition com.2** (Saturated set). A set  $\Gamma$  of formulas of a language  $\mathcal{L}$  is saturated iff for each formula  $\varphi(x) \in \text{Frm}(\mathcal{L})$  with one free variable  $x$  there is a constant symbol  $c \in \mathcal{L}$  such that  $\exists x \varphi(x) \rightarrow \varphi(c) \in \Gamma$ .

The following definition will be used in the proof of the next theorem.

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defn:henkin-exp

**Definition com.3.** Let  $\mathcal{L}'$  be as in Proposition com.1. Fix an enumeration  $\varphi_0(x_0), \varphi_1(x_1), \dots$  of all formulas  $\varphi_i(x_i)$  of  $\mathcal{L}'$  in which one variable ( $x_i$ ) occurs free. We define the sentences  $\theta_n$  by induction on  $n$ .

Let  $c_0$  be the first constant symbol among the  $d_i$  we added to  $\mathcal{L}$  which does not occur in  $\varphi_0(x_0)$ . Assuming that  $\theta_0, \dots, \theta_{n-1}$  have already been defined, let  $c_n$  be the first among the new constant symbols  $d_i$  that occurs neither in  $\theta_0, \dots, \theta_{n-1}$  nor in  $\varphi_n(x_n)$ .

Now let  $\theta_n$  be the formula  $\exists x_n \varphi_n(x_n) \rightarrow \varphi_n(c_n)$ .

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lem:henkin

**Lemma com.4.** *Every consistent set  $\Gamma$  can be extended to a saturated consistent set  $\Gamma'$ .*

*Proof.* Given a consistent set of sentences  $\Gamma$  in a language  $\mathcal{L}$ , expand the language by adding a denumerable set of new constant symbols to form  $\mathcal{L}'$ . By Proposition com.1,  $\Gamma$  is still consistent in the richer language. Further, let  $\theta_i$  be as in Definition com.3. Let

$$\begin{aligned}\Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \Gamma_n \cup \{\theta_n\}\end{aligned}$$

i.e.,  $\Gamma_{n+1} = \Gamma \cup \{\theta_0, \dots, \theta_n\}$ , and let  $\Gamma' = \bigcup_n \Gamma_n$ .  $\Gamma'$  is clearly saturated.

If  $\Gamma'$  were inconsistent, then for some  $n$ ,  $\Gamma_n$  would be inconsistent (Exercise: explain why). So to show that  $\Gamma'$  is consistent it suffices to show, by induction on  $n$ , that each set  $\Gamma_n$  is consistent.

The induction basis is simply the claim that  $\Gamma_0 = \Gamma$  is consistent, which is the hypothesis of the theorem. For the induction step, suppose that  $\Gamma_n$  is consistent but  $\Gamma_{n+1} = \Gamma_n \cup \{\theta_n\}$  is inconsistent. Recall that  $\theta_n$  is  $\exists x_n \varphi_n(x_n) \rightarrow$

$\varphi_n(c_n)$ , where  $\varphi_n(x_n)$  is a formula of  $\mathcal{L}'$  with only the variable  $x_n$  free. By the way we've chosen the  $c_n$  (see Definition com.3),  $c_n$  does not occur in  $A_n(x_n)$  nor in  $\Gamma_n$ .

If  $\Gamma_n \cup \{\theta_n\}$  is inconsistent, then  $\Gamma_n \vdash \neg\theta_n$ , and hence both of the following hold:

$$\Gamma_n \vdash \exists x_n \varphi_n(x_n) \quad \Gamma_n \vdash \neg\varphi_n(c_n)$$

Since  $c_n$  does not occur in  $\Gamma_n$  or in  $\varphi_n(x_n)$ ,  $\text{??????}$  applies. From  $\Gamma_n \vdash \neg\varphi_n(c_n)$ , we obtain  $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$ . Thus we have that both  $\Gamma_n \vdash \exists x_n \varphi_n$  and  $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$ , so  $\Gamma_n$  itself is inconsistent. (Note that  $\forall x_n \neg\varphi_n(x_n) \vdash \neg\exists x_n \varphi_n(x_n)$ .) Contradiction:  $\Gamma_n$  was supposed to be consistent. Hence  $\Gamma_n \cup \{\theta_n\}$  is consistent.  $\square$

explanation

We'll now show that *complete*, consistent sets which are saturated have the property that it contains a universally quantified sentence iff it contains all its instances and it contains an existentially quantified sentence iff it contains at least one instance. We'll use this to show that the *structure* we'll generate from a complete, consistent, saturated set makes all its quantified sentences true.

**Proposition com.5.** *Suppose  $\Gamma$  is complete, consistent, and saturated.*

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prop:saturated-instances

1.  $\exists x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for at least one closed term  $t$ .
2.  $\forall x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for all closed terms  $t$ .

*Proof.* 1. First suppose that  $\exists x \varphi(x) \in \Gamma$ . Because  $\Gamma$  is saturated,  $(\exists x \varphi(x) \rightarrow \varphi(c)) \in \Gamma$  for some constant symbol  $c$ . By  $\text{??????}$ , item (1), and  $\text{????}$ ,  $\varphi(c) \in \Gamma$ .

For the other direction, saturation is not necessary: Suppose  $\varphi(t) \in \Gamma$ . Then  $\Gamma \vdash \exists x \varphi(x)$  by  $\text{??????}$ , item (1). By  $\text{????}$ ,  $\exists x \varphi(x) \in \Gamma$ .

2. Suppose that  $\varphi(t) \in \Gamma$  for all closed terms  $t$ . By way of contradiction, assume  $\forall x \varphi(x) \notin \Gamma$ . Since  $\Gamma$  is complete,  $\neg\forall x \varphi(x) \in \Gamma$ . By saturation,  $(\exists x \neg\varphi(x) \rightarrow \neg\varphi(c)) \in \Gamma$  for some constant symbol  $c$ . By assumption, since  $c$  is a closed term,  $\varphi(c) \in \Gamma$ . But this would make  $\Gamma$  inconsistent. (Exercise: give the derivation that shows

$$\neg\forall x \varphi(x), \exists x \neg\varphi(x) \rightarrow \neg\varphi(c), \varphi(c)$$

is inconsistent.)

For the reverse direction, we do not need saturation: Suppose  $\forall x \varphi(x) \in \Gamma$ . Then  $\Gamma \vdash \varphi(t)$  by  $\text{??????}$ , item (2). We get  $\varphi(t) \in \Gamma$  by  $\text{??}$ .  $\square$

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**Bibliography**