

com.1 Henkin Expansion

fol:com:hen:sec Part of the challenge in proving the completeness theorem is that the model explanation we construct from a complete consistent set Γ must make all the quantified **formulas** in Γ true. In order to guarantee this, we use a trick due to Leon Henkin. In essence, the trick consists in expanding the language by infinitely many **constant symbols** and adding, for each **formula** with one free **variable** $\varphi(x)$ a formula of the form $\exists x \varphi \rightarrow \varphi(c)$, where c is one of the new **constant symbols**. When we construct the **structure** satisfying Γ , this will guarantee that each true existential sentence has a witness among the new constants.

fol:com:hen:prop:lang-exp **Proposition com.1.** *If Γ is consistent in \mathcal{L} and \mathcal{L}' is obtained from \mathcal{L} by adding a denumerable set of new **constant symbols** d_0, d_1, \dots , then Γ is consistent in \mathcal{L}' .*

Definition com.2 (Saturated set). A set Γ of **formulas** of a language \mathcal{L} is *saturated* iff for each **formula** $\varphi(x) \in \text{Frm}(\mathcal{L})$ with one free **variable** x there is a **constant symbol** $c \in \mathcal{L}$ such that $\exists x \varphi(x) \rightarrow \varphi(c) \in \Gamma$.

The following definition will be used in the proof of the next theorem.

fol:com:hen:defn:henkin-exp **Definition com.3.** Let \mathcal{L}' be as in [Proposition com.1](#). Fix an enumeration $\varphi_0(x_0), \varphi_1(x_1), \dots$ of all **formulas** $\varphi_i(x_i)$ of \mathcal{L}' in which one variable (x_i) occurs free. We define the **sentences** θ_n by induction on n .

Let c_0 be the first **constant symbol** among the d_i we added to \mathcal{L} which does not occur in $\varphi_0(x_0)$. Assuming that $\theta_0, \dots, \theta_{n-1}$ have already been defined, let c_n be the first among the new **constant symbols** d_i that occurs neither in $\theta_0, \dots, \theta_{n-1}$ nor in $\varphi_n(x_n)$.

Now let θ_n be the **formula** $\exists x_n \varphi_n(x_n) \rightarrow \varphi_n(c_n)$.

fol:com:hen:lem:henkin **Lemma com.4.** *Every consistent set Γ can be extended to a saturated consistent set Γ' .*

Proof. Given a consistent set of sentences Γ in a language \mathcal{L} , expand the language by adding a **denumerable** set of new **constant symbols** to form \mathcal{L}' . By [Proposition com.1](#), Γ is still consistent in the richer language. Further, let θ_i be as in [Definition com.3](#). Let

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \Gamma_n \cup \{\theta_n\} \end{aligned}$$

i.e., $\Gamma_{n+1} = \Gamma \cup \{\theta_0, \dots, \theta_n\}$, and let $\Gamma' = \bigcup_n \Gamma_n$. Γ' is clearly saturated.

If Γ' were inconsistent, then for some n , Γ_n would be inconsistent (Exercise: explain why). So to show that Γ' is consistent it suffices to show, by induction on n , that each set Γ_n is consistent.

The induction basis is simply the claim that $\Gamma_0 = \Gamma$ is consistent, which is the hypothesis of the theorem. For the induction step, suppose that Γ_n is consistent but $\Gamma_{n+1} = \Gamma_n \cup \{\theta_n\}$ is inconsistent. Recall that θ_n is $\exists x_n \varphi_n(x_n) \rightarrow$

$\varphi_n(c_n)$, where $\varphi_n(x_n)$ is a formula of \mathcal{L}' with only the variable x_n free. By the way we've chosen the c_n (see Definition com.3), c_n does not occur in $A_n(x_n)$ nor in Γ_n .

If $\Gamma_n \cup \{\theta_n\}$ is inconsistent, then $\Gamma_n \vdash \neg\theta_n$, and hence both of the following hold:

$$\Gamma_n \vdash \exists x_n \varphi_n(x_n) \quad \Gamma_n \vdash \neg\varphi_n(c_n)$$

Since c_n does not occur in Γ_n or in $\varphi_n(x_n)$, $\forall x_n \neg\varphi_n(x_n)$ applies. From $\Gamma_n \vdash \neg\varphi_n(c_n)$, we obtain $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$. Thus we have that both $\Gamma_n \vdash \exists x_n \varphi_n$ and $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$, so Γ_n itself is inconsistent. (Note that $\forall x_n \neg\varphi_n(x_n) \vdash \neg\exists x_n \varphi_n(x_n)$.) Contradiction: Γ_n was supposed to be consistent. Hence $\Gamma_n \cup \{\theta_n\}$ is consistent. \square

explanation

We'll now show that *complete*, consistent sets which are saturated have the property that it contains a universally quantified sentence iff it contains all its instances and it contains an existentially quantified sentence iff it contains at least one instance. We'll use this to show that the *structure* we'll generate from a complete, consistent, saturated set makes all its quantified sentences true.

Proposition com.5. *Suppose Γ is complete, consistent, and saturated.*

fol:com:hen:
prop:saturated-instances

1. $\exists x \varphi(x) \in \Gamma$ iff $\varphi(t) \in \Gamma$ for at least one closed term t .
2. $\forall x \varphi(x) \in \Gamma$ iff $\varphi(t) \in \Gamma$ for all closed terms t .

Proof. 1. First suppose that $\exists x \varphi(x) \in \Gamma$. Because Γ is saturated, $(\exists x \varphi(x) \rightarrow \varphi(c)) \in \Gamma$ for some constant symbol c . By $\forall x (\varphi(x) \rightarrow \varphi(c))$, item (1), and $\exists x \varphi(x)$, $\varphi(c) \in \Gamma$.

For the other direction, saturation is not necessary: Suppose $\varphi(t) \in \Gamma$. Then $\Gamma \vdash \exists x \varphi(x)$ by \exists -introduction, item (1). By \exists -elimination, $\exists x \varphi(x) \in \Gamma$.

2. Suppose that $\varphi(t) \in \Gamma$ for all closed terms t . By way of contradiction, assume $\forall x \varphi(x) \notin \Gamma$. Since Γ is complete, $\neg\forall x \varphi(x) \in \Gamma$. By saturation, $(\exists x \neg\varphi(x) \rightarrow \neg\varphi(c)) \in \Gamma$ for some constant symbol c . By assumption, since c is a closed term, $\varphi(c) \in \Gamma$. But this would make Γ inconsistent. (Exercise: give the derivation that shows

$$\neg\forall x \varphi(x), \exists x \neg\varphi(x) \rightarrow \neg\varphi(c), \varphi(c)$$

is inconsistent.)

For the reverse direction, we do not need saturation: Suppose $\forall x \varphi(x) \in \Gamma$. Then $\Gamma \vdash \varphi(t)$ by \forall -elimination, item (2). We get $\varphi(t) \in \Gamma$ by \exists -introduction. \square

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Bibliography