The Löwenheim-Skolem Theorem

The Löwenheim-Skolem Theorem says that if a theory has an infinite model, then it also has a model that is at most denumerable. An immediate consequence of this fact is that first-order logic cannot express that the size of a structure is non-enumerable: any sentence or set of sentences satisfied in all non-enumerable structures is also satisfied in some enumerable structure.

Theorem com.1. If $\Gamma$ is consistent then it has an enumerable model, i.e., it is satisfiable in a structure whose domain is either finite or denumerable.

Proof. If $\Gamma$ is consistent, the structure $\mathcal{M}$ delivered by the proof of the completeness theorem has a domain $|\mathcal{M}|$ that is no larger than the set of the terms of the language $\mathcal{L}$. So $\mathcal{M}$ is at most denumerable. □

Theorem com.2. If $\Gamma$ is consistent set of sentences in the language of first-order logic without identity, then it has a denumerable model, i.e., it is satisfiable in a structure whose domain is infinite and enumerable.

Proof. If $\Gamma$ is consistent and contains no sentences in which identity appears, then the structure $\mathcal{M}$ delivered by the proof of the completeness theorem has a domain $|\mathcal{M}|$ identical to the set of terms of the language $\mathcal{L}'$. So $\mathcal{M}$ is denumerable, since Trm($\mathcal{L}'$) is. □

Example com.3 (Skolem’s Paradox). Zermelo-Fraenkel set theory ZFC is a very powerful framework in which practically all mathematical statements can be expressed, including facts about the sizes of sets. So for instance, ZFC can prove that the set $\mathbb{R}$ of real numbers is non-enumerable, it can prove Cantor’s Theorem that the power set of any set is larger than the set itself, etc. If ZFC is consistent, its models are all infinite, and moreover, they all contain elements about which the theory says that they are non-enumerable, such as the element that makes true the theorem of ZFC that the power set of the natural numbers exists. By the Löwenheim-Skolem Theorem, ZFC also has enumerable models—models that contain “non-enumerable” sets but which themselves are enumerable.

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Bibliography