

com.1 The Löwenheim-Skolem Theorem

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sec The Löwenheim-Skolem Theorem says that if a theory has an infinite model, then it also has a model that is at most **denumerable**. An immediate consequence of this fact is that first-order logic cannot express that the size of a structure is **non-enumerable**: any **sentence** or set of **sentences** satisfied in all **non-enumerable structures** is also satisfied in some **enumerable** structure.

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thm:downward-ls **Theorem com.1.** *If Γ is consistent then it has an **enumerable** model, i.e., it is satisfiable in a **structure** whose domain is either finite or **denumerable**.*

Proof. If Γ is consistent, the **structure** \mathfrak{M} delivered by the proof of the completeness theorem has a domain $|\mathfrak{M}|$ that is no larger than the set of the terms of the language \mathcal{L} . So \mathfrak{M} is at most **denumerable**. \square

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noidentity-ls **Theorem com.2.** *If Γ is a consistent set of **sentences** in the language of first-order logic without identity, then it has a **denumerable** model, i.e., it is satisfiable in a **structure** whose domain is infinite and **enumerable**.*

Proof. If Γ is consistent and contains no sentences in which identity appears, then the **structure** \mathfrak{M} delivered by the proof of the completeness theorem has a domain $|\mathfrak{M}|$ identical to the set of terms of the language \mathcal{L}' . So \mathfrak{M} is **denumerable**, since $\text{Trm}(\mathcal{L}')$ is. \square

Example com.3 (Skolem’s Paradox). Zermelo-Fraenkel set theory **ZFC** is a very powerful framework in which practically all mathematical statements can be expressed, including facts about the sizes of sets. So for instance, **ZFC** can prove that the set \mathbb{R} of real numbers is **non-enumerable**, it can prove Cantor’s Theorem that the power set of any set is larger than the set itself, etc. If **ZFC** is consistent, its models are all infinite, and moreover, they all contain **elements** about which the theory says that they are **non-enumerable**, such as the element that makes true the theorem of **ZFC** that the power set of the natural numbers exists. By the Löwenheim-Skolem Theorem, **ZFC** also has **enumerable** models—models that contain “**non-enumerable**” sets but which themselves are **enumerable**.

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Bibliography