com.1 The Löwenheim-Skolem Theorem

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The Löwenheim-Skolem Theorem says that if a theory has an infinite model, then it also has a model that is at most denumerable. An immediate consequene of this fact is that first-order logic cannot express that the size of a structure is non-enumerable: any sentence or set of sentences satisfied in all non-enumerable structures is also satisfied in some enumerable structure.

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Theorem com.1. *If Γ is consistent then it has an enumerable model, i.e., it is satisfiable in a structure whose domain is either finite or denumerable.*

Proof. If Γ is consistent, the structure $\mathfrak{M}$ delivered by the proof of the completeness theorem has a domain $|\mathfrak{M}|$ that is no larger than the set of the terms of the language $\mathcal{L}$. So $\mathfrak{M}$ is at most denumerable. □

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Theorem com.2. *If Γ is consistent set of sentences in the language of first-order logic without identity, then it has a denumerable model, i.e., it is satisfiable in a structure whose domain is infinite and enumerable.*

Proof. If Γ is consistent and contains no sentences in which identity appears, then the structure $\mathfrak{M}$ delivered by the proof of the completness theorem has a domain $|\mathfrak{M}|$ identical to the set of terms of the language $\mathcal{L}'$. So $\mathfrak{M}$ is denumerable, since $\mathrm{Trm}(\mathcal{L}')$ is. □

Example com.3 (Skolem's Paradox)**.** Zermelo-Fraenkel set theory **ZFC** is a very powerful framework in which practically all mathematical statements can be expressed, including facts about the sizes of sets. So for instance, **ZFC** can prove that the set $\mathbb{R}$ of real numbers is non-enumerable, it can prove Cantor's Theorem that the power set of any set is larger than the set itself, etc. If **ZFC** is consistent, its models are all infinite, and moreover, they all contain elements about which the theory says that they are non-enumerable, such as the element that makes true the theorem of **ZFC** that the power set of the natural numbers exists. By the Löwenheim-Skolem Theorem, **ZFC** also has enumerable models—models that contain "non-enumerable" sets but which themselves are enumerable.

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Bibliography