

## com.1 Construction of a Model

fol:com:mod:  
sec

Right now we are not concerned about  $=$ , i.e., we only want to show that a consistent set  $\Gamma$  of sentences not containing  $=$  is satisfiable. We first extend  $\Gamma$  to a consistent, complete, and saturated set  $\Gamma^*$ . In this case, the definition of a model  $\mathfrak{M}(\Gamma^*)$  is simple: We take the set of closed terms of  $\mathcal{L}'$  as the domain. We assign every constant symbol to itself, and make sure that more generally, for every closed term  $t$ ,  $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$ . The predicate symbols are assigned extensions in such a way that an atomic sentence is true in  $\mathfrak{M}(\Gamma^*)$  iff it is in  $\Gamma^*$ . This will obviously make all the atomic sentences in  $\Gamma^*$  true in  $\mathfrak{M}(\Gamma^*)$ . The rest are true provided the  $\Gamma^*$  we start with is consistent, complete, and saturated.

fol:com:mod:  
defn:termmodel

**Definition com.1** (Term model). Let  $\Gamma^*$  be a complete and consistent, saturated set of sentences in a language  $\mathcal{L}$ . The term model  $\mathfrak{M}(\Gamma^*)$  of  $\Gamma^*$  is the structure defined as follows:

1. The domain  $|\mathfrak{M}(\Gamma^*)|$  is the set of all closed terms of  $\mathcal{L}$ .
2. The interpretation of a constant symbol  $c$  is  $c$  itself:  $c^{\mathfrak{M}(\Gamma^*)} = c$ .
3. The function symbol  $f$  is assigned the function which, given as arguments the closed terms  $t_1, \dots, t_n$ , has as value the closed term  $f(t_1, \dots, t_n)$ :

$$f^{\mathfrak{M}(\Gamma^*)}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$

4. If  $R$  is an  $n$ -place predicate symbol, then

$$\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)} \text{ iff } R(t_1, \dots, t_n) \in \Gamma^*.$$

A structure  $\mathfrak{M}$  may make an existentially quantified sentence  $\exists x \varphi(x)$  true without there being an instance  $\varphi(t)$  that it makes true. A structure  $\mathfrak{M}$  may make all instances  $\varphi(t)$  of a universally quantified sentence  $\forall x \varphi(x)$  true, without making  $\forall x \varphi(x)$  true. This is because in general not every element of  $|\mathfrak{M}|$  is the value of a closed term ( $\mathfrak{M}$  may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model  $\mathfrak{M}(\Gamma^*)$  this wouldn't be necessary—because it is covered. This is the content of the next result.

fol:com:mod:  
prop:quant-termmodel

**Proposition com.2.** Let  $\mathfrak{M}(\Gamma^*)$  be the term model of Definition com.1.

1.  $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$  iff  $\mathfrak{M} \models \varphi(t)$  for at least one term  $t$ .
2.  $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$  iff  $\mathfrak{M} \models \varphi(t)$  for all terms  $t$ .

*Proof.* 1. By ??,  $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$  iff for at least one variable assignment  $s$ ,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . As  $|\mathfrak{M}(\Gamma^*)|$  consists of the closed terms of  $\mathcal{L}$ , this is the case iff there is at least one closed term  $t$  such that  $s(x) = t$  and  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$  iff  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ , where  $s(x) = t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$  iff  $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ , since  $\varphi(t)$  is a sentence.

2. By ??,  $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$  iff for every variable assignment  $s$ ,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . Recall that  $|\mathfrak{M}(\Gamma^*)|$  consists of the closed terms of  $\mathcal{L}$ , so for every closed term  $t$ ,  $s(x) = t$  is such a variable assignment, and for any variable assignment,  $s(x)$  is some closed term  $t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$  iff  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ , where  $s(x) = t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$  iff  $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ , since  $\varphi(t)$  is a sentence.

□

**Problem com.1.** Complete the proof of [Proposition com.2](#).

**Lemma com.3** (Truth Lemma). *Suppose  $\varphi$  does not contain  $=$ . Then [fol.com.mod:lem.truth](#)  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\varphi \in \Gamma^*$ .*

*Proof.* We prove both directions simultaneously, and by induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ :  $\mathfrak{M}(\Gamma^*) \not\models \perp$  by definition of satisfaction. On the other hand,  $\perp \notin \Gamma^*$  since  $\Gamma^*$  is consistent.
2.  $\varphi \equiv \top$ :  $\mathfrak{M}(\Gamma^*) \models \top$  by definition of satisfaction. On the other hand,  $\top \in \Gamma^*$  since  $\Gamma^*$  is consistent and [complete](#), and  $\Gamma^* \vdash \top$ .
3.  $\varphi \equiv R(t_1, \dots, t_n)$ :  $\mathfrak{M}(\Gamma^*) \models R(t_1, \dots, t_n)$  iff  $\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)}$  (by the definition of satisfaction) iff  $R(t_1, \dots, t_n) \in \Gamma^*$  (by the construction of  $\mathfrak{M}(\Gamma^*)$ ).
4.  $\varphi \equiv \neg\psi$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\mathfrak{M}(\Gamma^*) \not\models \psi$  (by definition of satisfaction). By induction hypothesis,  $\mathfrak{M}(\Gamma^*) \not\models \psi$  iff  $\psi \notin \Gamma^*$ . Since  $\Gamma^*$  is consistent and [complete](#),  $\psi \notin \Gamma^*$  iff  $\neg\psi \in \Gamma^*$ .
5.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff we have both  $\mathfrak{M}(\Gamma^*) \models \psi$  and  $\mathfrak{M}(\Gamma^*) \models \chi$  (by definition of satisfaction) iff both  $\psi \in \Gamma^*$  and  $\chi \in \Gamma^*$  (by the induction hypothesis). By [????](#), this is the case iff  $(\psi \wedge \chi) \in \Gamma^*$ .
6.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff at  $\mathfrak{M}(\Gamma^*) \models \psi$  or  $\mathfrak{M}(\Gamma^*) \models \chi$  (by definition of satisfaction) iff  $\psi \in \Gamma^*$  or  $\chi \in \Gamma^*$  (by induction hypothesis). This is the case iff  $(\psi \vee \chi) \in \Gamma^*$  (by [????](#)).
7.  $\varphi \equiv \psi \rightarrow \chi$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\mathfrak{M}(\Gamma^*) \not\models \psi$  or  $\mathfrak{M}(\Gamma^*) \models \chi$  (by definition of satisfaction) iff  $\psi \notin \Gamma^*$  or  $\chi \in \Gamma^*$  (by induction hypothesis). This is the case iff  $(\psi \rightarrow \chi) \in \Gamma^*$  (by [????](#)).
8.  $\varphi \equiv \forall x \psi(x)$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\mathfrak{M}(\Gamma^*) \models \psi(t)$  for all terms  $t$  ([Proposition com.2](#)). By induction hypothesis, this is the case iff  $\psi(t) \in \Gamma^*$  for all terms  $t$ , by ??, this in turn is the case iff  $\forall x \varphi(x) \in \Gamma^*$ .
9.  $\varphi \equiv \exists x \psi(x)$ :  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\mathfrak{M}(\Gamma^*) \models \psi(t)$  for at least one term  $t$  ([Proposition com.2](#)). By induction hypothesis, this is the case iff  $\psi(t) \in \Gamma^*$  for at least one term  $t$ . By ??, this in turn is the case iff  $\exists x \varphi(x) \in \Gamma^*$ .

□

**Problem com.2.** Complete the proof of [Lemma com.3](#).

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**Bibliography**