com.1 Construction of a Model

fol:com:mod:sec

explanation

Right now we are not concerned about $=$, i.e., we only want to show that a consistent set Γ of sentences not containing $=$ is satisfiable. We first extend Γ to a consistent, complete, and saturated set Γ^*. In this case, the definition of a model $\mathfrak{M}(\Gamma^*)$ is simple: We take the set of closed terms of $\mathcal{L}'$ as the domain. We assign every constant symbol to itself, and make sure that more generally, for every closed term t, $\mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$. The predicate symbols are assigned extensions in such a way that an atomic sentence is true in $\mathfrak{M}(\Gamma^*)$ iff it is in Γ^*. This will obviously make all the atomic sentences in Γ^* true in $\mathfrak{M}(\Gamma^*)$. The rest are true provided the Γ^* we start with is consistent, complete, and saturated.

fol:com:mod:defn:termmodel

Definition com.1 (Term model). Let Γ^* be a complete and consistent, saturated set of sentences in a language $\mathcal{L}$. The *term model* $\mathfrak{M}(\Gamma^*)$ of Γ^* is the structure defined as follows:

1. The domain $|\mathfrak{M}(\Gamma^*)|$ is the set of all closed terms of $\mathcal{L}$.
2. The interpretation of a constant symbol c is c itself: $c^{\mathfrak{M}(\Gamma^*)} = c$.
3. The function symbol f is assigned the function which, given as arguments the closed terms $t_1, \ldots, t_n$, has as value the closed term $f(t_1, \ldots, t_n)$:
$$f^{\mathfrak{M}(\Gamma^*)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$
4. If R is an n-place predicate symbol, then
$$\langle t_1, \ldots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)} \text{ iff } R(t_1, \ldots, t_n) \in \Gamma^*.$$

We will now check that we indeed have $\mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.

fol:com:mod:lem:val-in-termmodel

Lemma com.2. *Let $\mathfrak{M}(\Gamma^*)$ be the term model of Definition com.1, then $\mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.*

Proof. The proof is by induction on t, where the base case, when t is a constant symbol, follows directly from the definition of the term model. For the induction step assume $t_1, \ldots, t_n$ are closed terms such that $\mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t_i) = t_i$ and that f is an n-ary function symbol. Then

$$\begin{aligned} \mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(f(t_1, \ldots, t_n)) &= f^{\mathfrak{M}(\Gamma^*)}(\mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t_1), \ldots, \mathrm{Val}^{\mathfrak{M}(\Gamma^*)}(t_n)) \\ &= f^{\mathfrak{M}(\Gamma^*)}(t_1, \ldots, t_n) \\ &= f(t_1, \ldots, t_n), \end{aligned}$$

and so by induction this holds for every closed term t. □

explanation A structure $\mathfrak{M}$ may make an existentially quantified sentence $\exists x\,\varphi(x)$ true without there being an instance $\varphi(t)$ that it makes true. A structure $\mathfrak{M}$ may make all instances $\varphi(t)$ of a universally quantified sentence $\forall x\,\varphi(x)$ true, without making $\forall x\,\varphi(x)$ true. This is because in general not every element of $|\mathfrak{M}|$ is the value of a closed term ($\mathfrak{M}$ may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model $\mathfrak{M}(\Gamma^*)$ this wouldn't be necessary—because it is covered. This is the content of the next result.

Proposition com.3. *Let $\mathfrak{M}(\Gamma^*)$ be the term model of Definition com.1.* fol:com:mod:prop:quant-termmodel

1. $\mathfrak{M}(\Gamma^*) \vDash \exists x\,\varphi(x)$ *iff* $\mathfrak{M}(\Gamma^*) \vDash \varphi(t)$ *for at least one closed term t.*

2. $\mathfrak{M}(\Gamma^*) \vDash \forall x\,\varphi(x)$ *iff* $\mathfrak{M}(\Gamma^*) \vDash \varphi(t)$ *for all closed terms t.*

Proof.
1. By **??**, $\mathfrak{M}(\Gamma^*) \vDash \exists x\,\varphi(x)$ iff for at least one variable assignment s, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(x)$. As $|\mathfrak{M}(\Gamma^*)|$ consists of the closed terms of $\mathcal{L}$, this is the case iff there is at least one closed term t such that $s(x) = t$ and $\mathfrak{M}(\Gamma^*), s \vDash \varphi(x)$. By **??**, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(x)$ iff $\mathfrak{M}(\Gamma^*), s \vDash \varphi(t)$, where $s(x) = t$. By **??**, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(t)$ iff $\mathfrak{M}(\Gamma^*) \vDash \varphi(t)$, since $\varphi(t)$ is a sentence.

2. By **??**, $\mathfrak{M}(\Gamma^*) \vDash \forall x\,\varphi(x)$ iff for every variable assignment s, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(x)$. Recall that $|\mathfrak{M}(\Gamma^*)|$ consists of the closed terms of $\mathcal{L}$, so for every closed term t, $s(x) = t$ is such a variable assignment, and for any variable assignment, $s(x)$ is some closed term t. By **??**, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(x)$ iff $\mathfrak{M}(\Gamma^*), s \vDash \varphi(t)$, where $s(x) = t$. By **??**, $\mathfrak{M}(\Gamma^*), s \vDash \varphi(t)$ iff $\mathfrak{M}(\Gamma^*) \vDash \varphi(t)$, since $\varphi(t)$ is a sentence. □

Lemma com.4 (Truth Lemma). *Suppose φ does not contain $=$. Then $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\varphi \in \Gamma^*$.* fol:com:mod:lem:truth

Proof. We prove both directions simultaneously, and by induction on φ.

1. $\varphi \equiv \bot$: $\mathfrak{M}(\Gamma^*) \nvDash \bot$ by definition of satisfaction. On the other hand, $\bot \notin \Gamma^*$ since Γ^* is consistent.

2. $\varphi \equiv \top$: $\mathfrak{M}(\Gamma^*) \vDash \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since Γ^* is consistent and complete, and $\Gamma^* \vdash \top$.

3. $\varphi \equiv R(t_1, \ldots, t_n)$: $\mathfrak{M}(\Gamma^*) \vDash R(t_1, \ldots, t_n)$ iff $\langle t_1, \ldots, t_n\rangle \in R^{\mathfrak{M}(\Gamma^*)}$ (by the definition of satisfaction) iff $R(t_1, \ldots, t_n) \in \Gamma^*$ (by the construction of $\mathfrak{M}(\Gamma^*)$).

4. $\varphi \equiv \neg\psi$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \nvDash \psi$ (by definition of satisfaction). By induction hypothesis, $\mathfrak{M}(\Gamma^*) \nvDash \psi$ iff $\psi \notin \Gamma^*$. Since Γ^* is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg\psi \in \Gamma^*$.

5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff we have both $\mathfrak{M}(\Gamma^*) \vDash \psi$ and $\mathfrak{M}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By **????**, this is the case iff $(\psi \wedge \chi) \in \Gamma^*$.

6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \vDash \psi$ or $\mathfrak{M}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \vee \chi) \in \Gamma^*$ (by **????**).

7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \nvDash \psi$ or $\mathfrak{M}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by **????**).

8. $\varphi \equiv \forall x\, \psi(x)$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \vDash \psi(t)$ for all terms t (Proposition com.3). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for all terms t, by **??**, this in turn is the case iff $\forall x\, \varphi(x) \in \Gamma^*$.

9. $\varphi \equiv \exists x\, \psi(x)$: $\mathfrak{M}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \vDash \psi(t)$ for at least one term t (Proposition com.3). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for at least one term t. By **??**, this in turn is the case iff $\exists x\, \psi(x) \in \Gamma^*$. □

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Bibliography