com.1 Construction of a Model

Right now we are not concerned about =, i.e., we only want to show that a consistent set \( \Gamma \) of sentences not containing = is satisfiable. We first extend \( \Gamma \) to a consistent, complete, and saturated set \( \Gamma^* \). In this case, the definition of a model \( \mathfrak{M}(\Gamma^*) \) is simple: We take the set of closed terms of \( \mathcal{L} \) as the domain. We assign every constant symbol to itself, and make sure that more generally, for every closed term \( t \), \( \text{Val}_{\mathfrak{M}}(\Gamma^*)(t) = t \). The predicate symbols are assigned extensions in such a way that an atomic sentence is true in \( \mathfrak{M}(\Gamma^*) \) iff it is in \( \Gamma^* \). This will obviously make all the atomic sentences in \( \Gamma^* \) true in \( \mathfrak{M}(\Gamma^*) \). The rest are true provided the \( \Gamma^* \) we start with is consistent, complete, and saturated.

**Definition com.1** (Term model). Let \( \Gamma^* \) be a complete and consistent, saturated set of sentences in a language \( \mathcal{L} \). The term model \( \mathfrak{M}(\Gamma^*) \) of \( \Gamma^* \) is the structure defined as follows:

1. The domain \( |\mathfrak{M}(\Gamma^*)| \) is the set of all closed terms of \( \mathcal{L} \).
2. The interpretation of a constant symbol \( c \) is \( c \) itself: \( \mathfrak{e}^{|\mathfrak{M}(\Gamma^*)|}(c) = c \).
3. The function symbol \( f \) is assigned the function which, given as arguments the closed terms \( t_1, \ldots, t_n \), has as value the closed term \( f(t_1, \ldots, t_n) \):
   \[
   f^{|\mathfrak{M}(\Gamma^*)|}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)
   \]
4. If \( R \) is an \( n \)-place predicate symbol, then
   \[
   (t_1, \ldots, t_n) \in R^{|\mathfrak{M}(\Gamma^*)|} \iff R(t_1, \ldots, t_n) \in \Gamma^*.
   \]

A structure \( \mathfrak{M} \) may make an existentially quantified sentence \( \exists x \varphi(x) \) true without there being an instance \( \varphi(t) \) that it makes true. A structure \( \mathfrak{M} \) may make all instances \( \varphi(t) \) of a universally quantified sentence \( \forall x \varphi(x) \) true, without making \( \forall x \varphi(x) \) true. This is because in general not every element of \( |\mathfrak{M}| \) is the value of a closed term (\( \mathfrak{M} \) may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model \( \mathfrak{M}(\Gamma^*) \) this wouldn’t be necessary—because it is covered. This is the content of the next result.

**Proposition com.2.** Let \( \mathfrak{M}(\Gamma^*) \) be the term model of Definition com.1.

1. \( \mathfrak{M}(\Gamma^*) \models \exists x \varphi(x) \iff \mathfrak{M} \models \varphi(t) \) for at least one term \( t \).
2. \( \mathfrak{M}(\Gamma^*) \models \forall x \varphi(x) \iff \mathfrak{M} \models \varphi(t) \) for all terms \( t \).

**Proof.**

1. By ??, \( \mathfrak{M}(\Gamma^*) \models \exists x \varphi(x) \iff \mathfrak{M} \models \varphi(t) \) for at least one variable assignment \( s \), \( \mathfrak{M}(\Gamma^*), s \models \varphi(x) \). As \( |\mathfrak{M}(\Gamma^*)| \) consists of the closed terms of \( \mathcal{L} \), this is the case iff there is at least one closed term \( t \) such that \( s(x) = t \) and \( \mathfrak{M}(\Gamma^*), s \models \varphi(x) \). By ??, \( \mathfrak{M}(\Gamma^*), s \models \varphi(x) \iff \mathfrak{M}(\Gamma^*), s \models \varphi(t) \), where \( s(x) = t \). By ??, \( \mathfrak{M}(\Gamma^*), s \models \varphi(t) \iff \mathfrak{M}(\Gamma^*) \models \varphi(t) \), since \( \varphi(t) \) is a sentence.
2. By ??, $\mathcal{M}(\Gamma^*) \vDash \forall x.\varphi(x)$ iff for every variable assignment $s$, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$. Recall that $|\mathcal{M}(\Gamma^*)|$ consists of the closed terms of $\mathcal{L}$, so for every closed term $t$, $s(x) = t$ is such a variable assignment, and for any variable assignment, $s(x)$ is some closed term $t$. By ??, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$ iff $\mathcal{M}(\Gamma^*), s \models \varphi(t)$, where $s(x) = t$. By ??, $\mathcal{M}(\Gamma^*), s \models \varphi(t)$ iff $\mathcal{M}(\Gamma^*) \models \varphi(t)$, since $\varphi(t)$ is a sentence.

\[ \square \]

\textbf{Lemma com.3} (Truth Lemma). Suppose $\varphi$ does not contain $\equiv$. Then $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$.

\textit{Proof}. We prove both directions simultaneously, and by induction on $\varphi$.

1. $\varphi \equiv \bot$: $\mathcal{M}(\Gamma^*) \not\models \bot$ by definition of satisfaction. On the other hand, $\bot \notin \Gamma^*$ since $\Gamma^*$ is consistent.

2. $\varphi \equiv \top$: $\mathcal{M}(\Gamma^*) \models \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since $\Gamma^*$ is consistent and complete, and $\Gamma^* \models \top$.

3. $\varphi \equiv \forall x.\chi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff we have both $\mathcal{M}(\Gamma^*) \models \psi$ and $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By ??, this is the case iff $(\psi \land \chi) \in \Gamma^*$.

4. $\varphi \equiv \neg \psi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \not\models \psi$ (by definition of satisfaction). By induction hypothesis, $\mathcal{M}(\Gamma^*) \not\models \psi$ iff $\psi \notin \Gamma^*$. Since $\Gamma^*$ is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg \psi \in \Gamma^*$.

5. $\varphi \equiv \chi \land \psi$: $\mathcal{M}(\Gamma^*) \models \varphi$ if we have both $\mathcal{M}(\Gamma^*) \models \psi$ and $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By ??, this is the case iff $(\psi \land \chi) \in \Gamma^*$.

6. $\varphi \equiv \chi \lor \psi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff at $\mathcal{M}(\Gamma^*) \models \psi$ or $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \lor \chi) \in \Gamma^*$ (by ??).

7. $\varphi \equiv \chi \rightarrow \psi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \not\models \psi$ or $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by ??).

8. $\varphi \equiv \forall x.\psi(x)$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \models \psi(t)$ for all terms $t$ (Proposition com.2). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for all terms $t$, by ??, this in turn is the case iff $\forall x.\varphi(x) \in \Gamma^*$.

9. $\varphi \equiv \exists x.\psi(x)$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \models \psi(t)$ for at least one term $t$ (Proposition com.2). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for at least one term $t$. By ??, this in turn is the case iff $\exists x.\varphi(x) \in \Gamma^*$.

\[ \square \]
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