**Com.1 Construction of a Model**

Right now we are not concerned about =, i.e., we only want to show that a consistent set \( I \) of sentences not containing = is satisfiable. We first extend \( I \) to a consistent, complete, and saturated set \( I^* \). In this case, the definition of a model \( M(I^*) \) is simple: We take the set of closed terms of \( L' \) as the domain. We assign every constant symbol to itself, and make sure that more generally, for every closed term \( t \), \( \text{Val}_M(I^*) = t \). The predicate symbols are assigned extensions in such a way that an atomic sentence is true in \( M(I^*) \) iff it is in \( I^* \). This will obviously make all the atomic sentences in \( I^* \) true in \( M(I^*) \). The rest are true provided the \( I^* \) we start with is consistent, complete, and saturated.

**Definition com.1 (Term model).** Let \( I^* \) be a complete and consistent, saturated set of sentences in a language \( L \). The term model \( M(I^*) \) of \( I^* \) is the structure defined as follows:

1. The domain \( |M(I^*)| \) is the set of all closed terms of \( L \).
2. The interpretation of a constant symbol \( c \) is \( c \) itself: \( c_{M(I^*)} = c \).
3. The function symbol \( f \) is assigned the function which, given as arguments the closed terms \( t_1, \ldots, t_n \), has as value the closed term \( f(t_1, \ldots, t_n) \):
   \[
   f_{M(I^*)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)
   \]
4. If \( R \) is an \( n \)-place predicate symbol, then
   \[
   \langle t_1, \ldots, t_n \rangle \in R_{M(I^*)} \iff R(t_1, \ldots, t_n) \in I^*.
   \]

A structure \( M \) may make an existentially quantified sentence \( \exists x \varphi(x) \) true without there being an instance \( \varphi(t) \) that it makes true. A structure \( M \) may make all instances \( \varphi(t) \) of a universally quantified sentence \( \forall x \varphi(x) \) true, without making \( \forall x \varphi(x) \) true. This is because in general not every element of \( |M| \) is the value of a closed term (\( M \) may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model \( M(I^*) \) this wouldn’t be necessary—because it is covered. This is the content of the next result.

**Proposition com.2.** Let \( M(I^*) \) be the term model of Definition com.1.

1. \( M(I^*) \models \exists x \varphi(x) \) iff \( M \models \varphi(s) \) for at least one term \( s \).
2. \( M(I^*) \models \forall x \varphi(x) \) iff \( M \models \varphi(t) \) for all terms \( t \).

**Proof.** 1. By ??, \( M(I^*) \models \exists x \varphi(x) \) iff for at least one variable assignment \( s, M(I^*), s \models \varphi(x) \). As \( |M(I^*)| \) consists of the closed terms of \( L \), this is the case iff there is at least one closed term \( t \) such that \( s(x) = t \) and \( M(I^*), s \models \varphi(x) \). By ??, \( M(I^*), s \models \varphi(x) \) iff \( M(I^*), s \models \varphi(t) \), where \( s(x) = t \). By ??, \( M(I^*), s \models \varphi(t) \) iff \( M(I^*), s \models \varphi(t) \), since \( \varphi(t) \) is a sentence.
Lemma com.3 (Truth Lemma). Suppose \( \varphi \) does not contain \( = \). Then \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \varphi \in \Gamma^* \).

Proof. We prove both directions simultaneously, and by induction on \( \varphi \).

1. \( \varphi \equiv \bot \): \( \mathcal{M}(\Gamma^*) \not\models \bot \) by definition of satisfaction. On the other hand, \( \bot \not\in \Gamma^* \) since \( \Gamma^* \) is consistent.

2. \( \varphi \equiv \top \): \( \mathcal{M}(\Gamma^*) \models \top \) by definition of satisfaction. On the other hand, \( \top \in \Gamma^* \) since \( \Gamma^* \) is consistent and complete, and \( \Gamma^* \vdash \top \).

3. \( \varphi \equiv R(t_1, \ldots, t_n); \mathcal{M}(\Gamma^*) \models R(t_1, \ldots, t_n) \) iff \( (t_1, \ldots, t_n) \in R^{\mathcal{M}(\Gamma^*)} \) (by the definition of satisfaction) iff \( R(t_1, \ldots, t_n) \in \Gamma^* \) (by the construction of \( \mathcal{M}(\Gamma^*) \)).

4. \( \varphi \equiv \neg \psi \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \mathcal{M}(\Gamma^*) \not\models \psi \) (by definition of satisfaction). By induction hypothesis, \( \mathcal{M}(\Gamma^*) \not\models \psi \) iff \( \psi \not\in \Gamma^* \). Since \( \Gamma^* \) is consistent and complete, \( \psi \not\in \Gamma^* \) iff \( \neg \psi \in \Gamma^* \).

5. \( \varphi \equiv \psi \land \chi \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff we have both \( \mathcal{M}(\Gamma^*) \models \psi \) and \( \mathcal{M}(\Gamma^*) \models \chi \) (by definition of satisfaction) iff both \( \psi \in \Gamma^* \) and \( \chi \in \Gamma^* \) (by the induction hypothesis). By \( ????, \) this is the case iff \( (\psi \land \chi) \in \Gamma^* \).

6. \( \varphi \equiv \psi \lor \chi \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \mathcal{M}(\Gamma^*) \models \psi \) or \( \mathcal{M}(\Gamma^*) \models \chi \) (by definition of satisfaction) iff \( \psi \in \Gamma^* \) or \( \chi \in \Gamma^* \) (by induction hypothesis). This is the case iff \( (\psi \lor \chi) \in \Gamma^* \) (by \( ????, \)).

7. \( \varphi \equiv \psi \rightarrow \chi \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \mathcal{M}(\Gamma^*) \not\models \psi \) or \( \mathcal{M}(\Gamma^*) \models \chi \) (by definition of satisfaction) iff \( \psi \not\in \Gamma^* \) or \( \chi \in \Gamma^* \) (by induction hypothesis). This is the case iff \( (\psi \rightarrow \chi) \in \Gamma^* \) (by \( ????, \)).

8. \( \varphi \equiv \forall x \psi(x) \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \mathcal{M}(\Gamma^*) \models \psi(t) \) for all terms \( t \) (Proposition com.2). By induction hypothesis, this is the case iff \( \psi(t) \in \Gamma^* \) for all terms \( t \), by \( ???, \) this in turn is the case iff \( \forall x \varphi(x) \in \Gamma^* \).

9. \( \varphi \equiv \exists x \psi(x) \): \( \mathcal{M}(\Gamma^*) \models \varphi \) iff \( \mathcal{M}(\Gamma^*) \models \psi(t) \) for at least one term \( t \) (Proposition com.2). By induction hypothesis, this is the case iff \( \psi(t) \in \Gamma^* \) for at least one term \( t \). By \( ???, \) this in turn is the case iff \( \exists x \varphi(x) \in \Gamma^* \).
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