

## Chapter udf

# The Completeness Theorem

### com.1 Introduction

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sec The completeness theorem is one of the most fundamental results about logic. It comes in two formulations, the equivalence of which we'll prove. In its first formulation it says something fundamental about the relationship between semantic consequence and our proof system: if a **sentence**  $\varphi$  follows from some **sentences**  $\Gamma$ , then there is also a **derivation** that establishes  $\Gamma \vdash \varphi$ . Thus, the proof system is as strong as it can possibly be without proving things that don't actually follow.

In its second formulation, it can be stated as a model existence result: every consistent set of **sentences** is satisfiable. Consistency is a proof-theoretic notion: it says that our proof system is unable to produce certain **derivations**. But who's to say that just because there are no **derivations** of a certain sort from  $\Gamma$ , it's guaranteed that there is a **structure**  $\mathfrak{M}$ ? Before the completeness theorem was first proved—in fact before we had the proof systems we now do—the great German mathematician David Hilbert held the view that consistency of mathematical theories guarantees the existence of the objects they are about. He put it as follows in a letter to Gottlob Frege:

If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Frege vehemently disagreed. The second formulation of the completeness theorem shows that Hilbert was right in at least the sense that if the axioms are consistent, then *some* **structure** exists that makes them all true.

These aren't the only reasons the completeness theorem—or rather, its proof—is important. It has a number of important consequences, some of which we'll discuss separately. For instance, since any **derivation** that shows  $\Gamma \vdash \varphi$  is finite and so can only use finitely many of the **sentences** in  $\Gamma$ , it follows by the completeness theorem that if  $\varphi$  is a consequence of  $\Gamma$ , it is already a

consequence of a finite subset of  $\Gamma$ . This is called *compactness*. Equivalently, if every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  itself must be consistent.

Although the compactness theorem follows from the completeness theorem via the detour through *derivations*, it is also possible to use the *the proof of the completeness theorem* to establish it directly. For what the proof does is take a set of *sentences* with a certain property—consistency—and constructs a *structure* out of this set that has certain properties (in this case, that it satisfies the set). Almost the very same construction can be used to directly establish compactness, by starting from “finitely satisfiable” sets of *sentences* instead of consistent ones. The construction also yields other consequences, e.g., that any satisfiable set of *sentences* has a finite or *denumerable* model. (This result is called the Löwenheim-Skolem theorem.) In general, the construction of *structures* from sets of *sentences* is used often in logic, and sometimes even in philosophy.

## com.2 Outline of the Proof

The proof of the completeness theorem is a bit complex, and upon first reading it, it is easy to get lost. So let us outline the proof. The first step is a shift of perspective, that allows us to see a route to a proof. When completeness is thought of as “whenever  $\Gamma \models \varphi$  then  $\Gamma \vdash \varphi$ ,” it may be hard to even come up with an idea: for to show that  $\Gamma \vdash \varphi$  we have to find a *derivation*, and it does not look like the hypothesis that  $\Gamma \models \varphi$  helps us for this in any way. For some proof systems it is possible to directly construct a *derivation*, but we will take a slightly different tack. The shift in perspective required is this: completeness can also be formulated as: “if  $\Gamma$  is consistent, it has a model.” Perhaps we can use the information in  $\Gamma$  together with the hypothesis that it is consistent to construct a model. After all, we know what kind of model we are looking for: one that is as  $\Gamma$  describes it!

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If  $\Gamma$  contains only atomic *sentences*, it is easy to construct a model for it. Suppose the atomic *sentences* are all of the form  $P(a_1, \dots, a_n)$  where the  $a_i$  are *constant symbols*. All we have to do is come up with a *domain*  $|\mathfrak{M}|$  and an assignment for  $P$  so that  $\mathfrak{M} \models P(a_1, \dots, a_n)$ . But that’s not very hard: put  $|\mathfrak{M}| = \mathbb{N}$ ,  $c_i^{\mathfrak{M}} = i$ , and for every  $P(a_1, \dots, a_n) \in \Gamma$ , put the tuple  $\langle k_1, \dots, k_n \rangle$  into  $P^{\mathfrak{M}}$ , where  $k_i$  is the index of the constant symbol  $a_i$  (i.e.,  $a_i \equiv c_{k_i}$ ).

Now suppose  $\Gamma$  contains some *formula*  $\neg\psi$ , with  $\psi$  atomic. We might worry that the construction of  $\mathfrak{M}$  interferes with the possibility of making  $\neg\psi$  true. But here’s where the consistency of  $\Gamma$  comes in: if  $\neg\psi \in \Gamma$ , then  $\psi \notin \Gamma$ , or else  $\Gamma$  would be inconsistent. And if  $\psi \notin \Gamma$ , then according to our construction of  $\mathfrak{M}$ ,  $\mathfrak{M} \not\models \psi$ , so  $\mathfrak{M} \models \neg\psi$ . So far so good.

What if  $\Gamma$  contains complex, non-atomic formulas? Say it contains  $\varphi \wedge \psi$ . To make that true, we should proceed as if both  $\varphi$  and  $\psi$  were in  $\Gamma$ . And if  $\varphi \vee \psi \in \Gamma$ , then we will have to make at least one of them true, i.e., proceed as if one of them was in  $\Gamma$ .

This suggests the following idea: we add additional **formulas** to  $\Gamma$  so as to (a) keep the resulting set consistent and (b) make sure that for every possible atomic **sentence**  $\varphi$ , either  $\varphi$  is in the resulting set, or  $\neg\varphi$  is, and (c) such that, whenever  $\varphi \wedge \psi$  is in the set, so are both  $\varphi$  and  $\psi$ , if  $\varphi \vee \psi$  is in the set, at least one of  $\varphi$  or  $\psi$  is also, etc. We keep doing this (potentially forever). Call the set of all **formulas** so added  $\Gamma^*$ . Then our construction above would provide us with a **structure**  $\mathfrak{M}$  for which we could prove, by induction, that all sentences in  $\Gamma^*$  are true in it, and hence also all sentence in  $\Gamma$  since  $\Gamma \subseteq \Gamma^*$ . It turns out that guaranteeing (a) and (b) is enough. A set of sentences for which (b) holds is called *complete*. So our task will be to extend the consistent set  $\Gamma$  to a consistent and complete set  $\Gamma^*$ .

There is one wrinkle in this plan: if  $\exists x \varphi(x) \in \Gamma$  we would hope to be able to pick some **constant symbol**  $c$  and add  $\varphi(c)$  in this process. But how do we know we can always do that? Perhaps we only have a few **constant symbols** in our language, and for each one of them we have  $\neg\varphi(c) \in \Gamma$ . We can't also add  $\varphi(c)$ , since this would make the set inconsistent, and we wouldn't know whether  $\mathfrak{M}$  has to make  $\varphi(c)$  or  $\neg\varphi(c)$  true. Moreover, it might happen that  $\Gamma$  contains only sentences in a language that has no constant symbols at all (e.g., the language of set theory).

The solution to this problem is to simply add infinitely many constants at the beginning, plus sentences that connect them with the quantifiers in the right way. (Of course, we have to verify that this cannot introduce an inconsistency.)

Our original construction works well if we only have **constant symbols** in the atomic sentences. But the language might also contain **function symbols**. In that case, it might be tricky to find the right functions on  $\mathbb{N}$  to assign to these **function symbols** to make everything work. So here's another trick: instead of using  $i$  to interpret  $c_i$ , just take the set of **constant symbols** itself as the domain. Then  $\mathfrak{M}$  can assign every **constant symbol** to itself:  $c_i^{\mathfrak{M}} = c_i$ . But why not go all the way: let  $|\mathfrak{M}|$  be all *terms* of the language! If we do this, there is an obvious assignment of functions (that take terms as arguments and have terms as values) to **function symbols**: we assign to the **function symbol**  $f_i^n$  the function which, given  $n$  terms  $t_1, \dots, t_n$  as input, produces the term  $f_i^n(t_1, \dots, t_n)$  as value.

The last piece of the puzzle is what to do with  $=$ . The **predicate symbol**  $=$  has a fixed interpretation:  $\mathfrak{M} \models t = t'$  iff  $\text{Val}^{\mathfrak{M}}(t) = \text{Val}^{\mathfrak{M}}(t')$ . Now if we set things up so that the **value** of a term  $t$  is  $t$  itself, then this **structure** will make *no* sentence of the form  $t = t'$  true unless  $t$  and  $t'$  are one and the same term. And of course this is a problem, since basically every interesting theory in a language with **function symbols** will have as theorems sentences  $t = t'$  where  $t$  and  $t'$  are not the same term (e.g., in theories of arithmetic:  $(0 + 0) = 0$ ). To solve this problem, we change the domain of  $\mathfrak{M}$ : instead of using terms as the objects in  $|\mathfrak{M}|$ , we use sets of terms, and each set is so that it contains all those terms which the sentences in  $\Gamma$  require to be equal. So, e.g., if  $\Gamma$  is a theory of arithmetic, one of these sets will contain:  $0, (0 + 0), (0 \times 0)$ , etc. This will be the set we assign to  $0$ , and it will turn out that this set is also the value of all the terms in it, e.g., also of  $(0 + 0)$ . Therefore, the sentence  $(0 + 0) = 0$  will

be true in this revised **structure**.

So here's what we'll do. First we investigate the properties of **complete** consistent sets, in particular we prove that a **complete** consistent set contains  $\varphi \wedge \psi$  iff it contains both  $\varphi$  and  $\psi$ ,  $\varphi \vee \psi$  iff it contains at least one of them, etc. (**Proposition com.2**). Then we define and investigate "saturated" sets of sentences. A saturated set is one which contains conditionals that link each quantified **sentence** to instances of it (**Definition com.5**). We show that any consistent set  $\Gamma$  can always be extended to a saturated set  $\Gamma'$  (**Lemma com.6**). If a set is consistent, saturated, and **complete** it also has the property that it contains  $\exists x \varphi(x)$  iff it contains  $\varphi(t)$  for some closed term  $t$  and  $\forall x \varphi(x)$  iff it contains  $\varphi(t)$  for all closed terms  $t$  (**Proposition com.7**). We'll then take the saturated consistent set  $\Gamma'$  and show that it can be extended to a saturated, consistent, and **complete** set  $\Gamma^*$  (**Lemma com.8**). This set  $\Gamma^*$  is what we'll use to define our term model  $\mathfrak{M}(\Gamma^*)$ . The term model has the set of closed terms as its domain, and the interpretation of its **predicate symbols** is given by the atomic **sentences** in  $\Gamma^*$  (**Definition com.9**). We'll use the properties of saturated, complete consistent sets to show that indeed  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\varphi \in \Gamma^*$  (**Lemma com.11**), and thus in particular,  $\mathfrak{M}(\Gamma^*) \models \Gamma$ . Finally, we'll consider how to define a term model if  $\Gamma$  contains = as well (**Definition com.15**) and show that it satisfies  $\Gamma^*$  (**Lemma com.17**).

### com.3 Complete Consistent Sets of Sentences

**Definition com.1** (Complete set). A set  $\Gamma$  of **sentences** is **complete** iff for any **sentence**  $\varphi$ , either  $\varphi \in \Gamma$  or  $\neg\varphi \in \Gamma$ .

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def:complete-set

explanation **Complete** sets of sentences leave no questions unanswered. For any **sentence**  $A$ ,  $\Gamma$  "says" if  $\varphi$  is true or false. The importance of **complete** sets extends beyond the proof of the completeness theorem. A theory which is **complete** and axiomatizable, for instance, is always decidable.

explanation **Complete** consistent sets are important in the completeness proof since we can guarantee that every consistent set of **sentences**  $\Gamma$  is contained in a **complete** consistent set  $\Gamma^*$ . A **complete** consistent set contains, for each **sentence**  $\varphi$ , either  $\varphi$  or its negation  $\neg\varphi$ , but not both. This is true in particular for atomic **sentences**, so from a **complete** consistent set in a language suitably expanded by **constant symbols**, we can construct a **structure** where the interpretation of **predicate symbols** is defined according to which atomic **sentences** are in  $\Gamma^*$ . This **structure** can then be shown to make all **sentences** in  $\Gamma^*$  (and hence also all those in  $\Gamma$ ) true. The proof of this latter fact requires that  $\neg\varphi \in \Gamma^*$  iff  $\varphi \notin \Gamma^*$ ,  $(\varphi \vee \psi) \in \Gamma^*$  iff  $\varphi \in \Gamma^*$  or  $\psi \in \Gamma^*$ , etc.

In what follows, we will often tacitly use the properties of reflexivity, monotonicity, and transitivity of  $\vdash$  (see ??????????????????).

**Proposition com.2.** *Suppose  $\Gamma$  is **complete** and consistent. Then:*

1. *If  $\Gamma \vdash \varphi$ , then  $\varphi \in \Gamma$ .*

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prop.ccs-and*

2.  $\varphi \wedge \psi \in \Gamma$  iff both  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ .

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3.  $\varphi \vee \psi \in \Gamma$  iff either  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ .

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4.  $\varphi \rightarrow \psi \in \Gamma$  iff either  $\varphi \notin \Gamma$  or  $\psi \in \Gamma$ .

*Proof.* Let us suppose for all of the following that  $\Gamma$  is **complete** and consistent.

1. If  $\Gamma \vdash \varphi$ , then  $\varphi \in \Gamma$ .

Suppose that  $\Gamma \vdash \varphi$ . Suppose to the contrary that  $\varphi \notin \Gamma$ . Since  $\Gamma$  is **complete**,  $\neg\varphi \in \Gamma$ . By **????????????????**,  $\Gamma$  is inconsistent. This contradicts the assumption that  $\Gamma$  is consistent. Hence, it cannot be the case that  $\varphi \notin \Gamma$ , so  $\varphi \in \Gamma$ .

2.  $\varphi \wedge \psi \in \Gamma$  iff both  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ :

For the forward direction, suppose  $\varphi \wedge \psi \in \Gamma$ . Then by **????????????????**, item (1),  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \psi$ . By (1),  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ , as required.

For the reverse direction, let  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ . By **????????????????**, item (2),  $\Gamma \vdash \varphi \wedge \psi$ . By (1),  $\varphi \wedge \psi \in \Gamma$ .

3. First we show that if  $\varphi \vee \psi \in \Gamma$ , then either  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ . Suppose  $\varphi \vee \psi \in \Gamma$  but  $\varphi \notin \Gamma$  and  $\psi \notin \Gamma$ . Since  $\Gamma$  is **complete**,  $\neg\varphi \in \Gamma$  and  $\neg\psi \in \Gamma$ . By **????????????????**, item (1),  $\Gamma$  is inconsistent, a contradiction. Hence, either  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ .

For the reverse direction, suppose that  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ . By **????????????????**, item (2),  $\Gamma \vdash \varphi \vee \psi$ . By (1),  $\varphi \vee \psi \in \Gamma$ , as required.

4. For the forward direction, suppose  $\varphi \rightarrow \psi \in \Gamma$ , and suppose to the contrary that  $\varphi \in \Gamma$  and  $\psi \notin \Gamma$ . On these assumptions,  $\varphi \rightarrow \psi \in \Gamma$  and  $\varphi \in \Gamma$ . By **????????????????**, item (1),  $\Gamma \vdash \psi$ . But then by (1),  $\psi \in \Gamma$ , contradicting the assumption that  $\psi \notin \Gamma$ .

For the reverse direction, first consider the case where  $\varphi \notin \Gamma$ . Since  $\Gamma$  is **complete**,  $\neg\varphi \in \Gamma$ . By **????????????????**, item (2),  $\Gamma \vdash \varphi \rightarrow \psi$ . Again by (1), we get that  $\varphi \rightarrow \psi \in \Gamma$ , as required.

Now consider the case where  $\psi \in \Gamma$ . By **????????????????**, item (2) again,  $\Gamma \vdash \varphi \rightarrow \psi$ . By (1),  $\varphi \rightarrow \psi \in \Gamma$ .

□

**Problem com.1.** Complete the proof of [Proposition com.2](#).

## com.4 Henkin Expansion

explanation Part of the challenge in proving the completeness theorem is that the model we construct from a complete consistent set  $\Gamma$  must make all the quantified **formulas** in  $\Gamma$  true. In order to guarantee this, we use a trick due to Leon Henkin. In essence, the trick consists in expanding the language by infinitely many **constant symbols** and adding, for each **formula** with one free **variable**  $\varphi(x)$  a formula of the form  $\exists x \varphi \rightarrow \varphi(c)$ , where  $c$  is one of the new **constant symbols**. When we construct the **structure** satisfying  $\Gamma$ , this will guarantee that each true existential sentence has a witness among the new constants. fol:com:hen:sec

**Proposition com.3.** *If  $\Gamma$  is consistent in  $\mathcal{L}$  and  $\mathcal{L}'$  is obtained from  $\mathcal{L}$  by adding a denumerable set of new **constant symbols**  $d_0, d_1, \dots$ , then  $\Gamma$  is consistent in  $\mathcal{L}'$ .* fol:com:hen:prop:lang-exp

**Definition com.4** (Saturated set). A set  $\Gamma$  of **formulas** of a language  $\mathcal{L}$  is *saturated* iff for each **formula**  $\varphi(x) \in \text{Frm}(\mathcal{L})$  with one free **variable**  $x$  there is a **constant symbol**  $c \in \mathcal{L}$  such that  $\exists x \varphi(x) \rightarrow \varphi(c) \in \Gamma$ .

The following definition will be used in the proof of the next theorem.

**Definition com.5.** Let  $\mathcal{L}'$  be as in [Proposition com.3](#). Fix an enumeration  $\varphi_0(x_0), \varphi_1(x_1), \dots$  of all **formulas**  $\varphi_i(x_i)$  of  $\mathcal{L}'$  in which one variable ( $x_i$ ) occurs free. We define the **sentences**  $\theta_n$  by induction on  $n$ . fol:com:hen:defn:henkin-exp

Let  $c_0$  be the first **constant symbol** among the  $d_i$  we added to  $\mathcal{L}$  which does not occur in  $\varphi_0(x_0)$ . Assuming that  $\theta_0, \dots, \theta_{n-1}$  have already been defined, let  $c_n$  be the first among the new **constant symbols**  $d_i$  that occurs neither in  $\theta_0, \dots, \theta_{n-1}$  nor in  $\varphi_n(x_n)$ .

Now let  $\theta_n$  be the **formula**  $\exists x_n \varphi_n(x_n) \rightarrow \varphi_n(c_n)$ .

**Lemma com.6.** *Every consistent set  $\Gamma$  can be extended to a saturated consistent set  $\Gamma'$ .* fol:com:hen:lem:henkin

*Proof.* Given a consistent set of sentences  $\Gamma$  in a language  $\mathcal{L}$ , expand the language by adding a denumerable set of new **constant symbols** to form  $\mathcal{L}'$ . By [Proposition com.3](#),  $\Gamma$  is still consistent in the richer language. Further, let  $\theta_i$  be as in [Definition com.5](#). Let

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \Gamma_n \cup \{\theta_n\} \end{aligned}$$

i.e.,  $\Gamma_{n+1} = \Gamma \cup \{\theta_0, \dots, \theta_n\}$ , and let  $\Gamma' = \bigcup_n \Gamma_n$ .  $\Gamma'$  is clearly saturated.

If  $\Gamma'$  were inconsistent, then for some  $n$ ,  $\Gamma_n$  would be inconsistent (Exercise: explain why). So to show that  $\Gamma'$  is consistent it suffices to show, by induction on  $n$ , that each set  $\Gamma_n$  is consistent.

The induction basis is simply the claim that  $\Gamma_0 = \Gamma$  is consistent, which is the hypothesis of the theorem. For the induction step, suppose that  $\Gamma_n$  is consistent but  $\Gamma_{n+1} = \Gamma_n \cup \{\theta_n\}$  is inconsistent. Recall that  $\theta_n$  is  $\exists x_n \varphi_n(x_n) \rightarrow$

$\varphi_n(c_n)$ , where  $\varphi_n(x_n)$  is a formula of  $\mathcal{L}'$  with only the variable  $x_n$  free. By the way we've chosen the  $c_n$  (see Definition com.5),  $c_n$  does not occur in  $A_n(x_n)$  nor in  $\Gamma_n$ .

If  $\Gamma_n \cup \{\theta_n\}$  is inconsistent, then  $\Gamma_n \vdash \neg\theta_n$ , and hence both of the following hold:

$$\Gamma_n \vdash \exists x_n \varphi_n(x_n) \quad \Gamma_n \vdash \neg\varphi_n(c_n)$$

Since  $c_n$  does not occur in  $\Gamma_n$  or in  $\varphi_n(x_n)$ ,  $\forall x_n \neg\varphi_n(x_n)$  applies. From  $\Gamma_n \vdash \neg\varphi_n(c_n)$ , we obtain  $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$ . Thus we have that both  $\Gamma_n \vdash \exists x_n \varphi_n$  and  $\Gamma_n \vdash \forall x_n \neg\varphi_n(x_n)$ , so  $\Gamma_n$  itself is inconsistent. (Note that  $\forall x_n \neg\varphi_n(x_n) \vdash \neg\exists x_n \varphi_n(x_n)$ .) Contradiction:  $\Gamma_n$  was supposed to be consistent. Hence  $\Gamma_n \cup \{\theta_n\}$  is consistent.  $\square$

We'll now show that *complete*, consistent sets which are saturated have the property that it contains a universally quantified sentence iff it contains all its instances and it contains an existentially quantified sentence iff it contains at least one instance. We'll use this to show that the structure we'll generate from a complete, consistent, saturated set makes all its quantified sentences true. explanation

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prop:saturated-instances*

**Proposition com.7.** *Suppose  $\Gamma$  is complete, consistent, and saturated.*

1.  $\exists x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for at least one closed term  $t$ .
2.  $\forall x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for all closed terms  $t$ .

*Proof.* 1. First suppose that  $\exists x \varphi(x) \in \Gamma$ . Because  $\Gamma$  is saturated,  $(\exists x \varphi(x) \rightarrow \varphi(c)) \in \Gamma$  for some constant symbol  $c$ . By  $\forall$ -introduction, item (1), and Proposition com.2(1),  $\varphi(c) \in \Gamma$ .

For the other direction, saturation is not necessary: Suppose  $\varphi(t) \in \Gamma$ . Then  $\Gamma \vdash \exists x \varphi(x)$  by  $\exists$ -introduction, item (1). By Proposition com.2(1),  $\exists x \varphi(x) \in \Gamma$ .

2. Suppose that  $\varphi(t) \in \Gamma$  for all closed terms  $t$ . By way of contradiction, assume  $\forall x \varphi(x) \notin \Gamma$ . Since  $\Gamma$  is complete,  $\neg\forall x \varphi(x) \in \Gamma$ . By saturation,  $(\exists x \neg\varphi(x) \rightarrow \neg\varphi(c)) \in \Gamma$  for some constant symbol  $c$ . By assumption, since  $c$  is a closed term,  $\varphi(c) \in \Gamma$ . But this would make  $\Gamma$  inconsistent. (Exercise: give the derivation that shows

$$\neg\forall x \varphi(x), \exists x \neg\varphi(x) \rightarrow \neg\varphi(c), \varphi(c)$$

is inconsistent.)

For the reverse direction, we do not need saturation: Suppose  $\forall x \varphi(x) \in \Gamma$ . Then  $\Gamma \vdash \varphi(t)$  by  $\forall$ -elimination, item (2). We get  $\varphi(t) \in \Gamma$  by Proposition com.2.  $\square$

## com.5 Lindenbaum's Lemma

explanation We now prove a lemma that shows that any consistent set of **sentences** is contained in some set of sentences which is not just consistent, but also **complete**. The proof works by adding one **sentence** at a time, guaranteeing at each step that the set remains consistent. We do this so that for every  $\varphi$ , either  $\varphi$  or  $\neg\varphi$  gets added at some stage. The union of all stages in that construction then contains either  $\varphi$  or its negation  $\neg\varphi$  and is thus complete. It is also consistent, since we made sure at each stage not to introduce an inconsistency. fol:com:lin:sec

**Lemma com.8** (Lindenbaum's Lemma). *Every consistent set  $\Gamma$  in a language  $\mathcal{L}$  can be extended to a **complete** and consistent set  $\Gamma^*$ .* fol:com:lin:lem:lindenbaum

*Proof.* Let  $\Gamma$  be consistent. Let  $\varphi_0, \varphi_1, \dots$  be an enumeration of all the **sentences** of  $\mathcal{L}$ . Define  $\Gamma_0 = \Gamma$ , and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let  $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$ .

Each  $\Gamma_n$  is consistent:  $\Gamma_0$  is consistent by definition. If  $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ , this is because the latter is consistent. If it isn't,  $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$ . We have to verify that  $\Gamma_n \cup \{\neg\varphi_n\}$  is consistent. Suppose it's not. Then *both*  $\Gamma_n \cup \{\varphi_n\}$  and  $\Gamma_n \cup \{\neg\varphi_n\}$  are inconsistent. This means that  $\Gamma_n$  would be inconsistent by **????????????????**, contrary to the induction hypothesis.

For every  $n$  and every  $i < n$ ,  $\Gamma_i \subseteq \Gamma_n$ . This follows by a simple induction on  $n$ . For  $n = 0$ , there are no  $i < 0$ , so the claim holds automatically. For the inductive step, suppose it is true for  $n$ . We have  $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$  or  $= \Gamma_n \cup \{\neg\varphi_n\}$  by construction. So  $\Gamma_n \subseteq \Gamma_{n+1}$ . If  $i < n$ , then  $\Gamma_i \subseteq \Gamma_n$  by inductive hypothesis, and so  $\subseteq \Gamma_{n+1}$  by transitivity of  $\subseteq$ .

From this it follows that every finite subset of  $\Gamma^*$  is a subset of  $\Gamma_n$  for some  $n$ , since each  $\psi \in \Gamma^*$  not already in  $\Gamma_0$  is added at some stage  $i$ . If  $n$  is the last one of these, then all  $\psi$  in the finite subset are in  $\Gamma_n$ . So, every finite subset of  $\Gamma^*$  is consistent. By **????????????????**,  $\Gamma^*$  is consistent.

Every **sentence** of  $\text{Frm}(\mathcal{L})$  appears on the list used to define  $\Gamma^*$ . If  $\varphi_n \notin \Gamma^*$ , then that is because  $\Gamma_n \cup \{\varphi_n\}$  was inconsistent. But then  $\neg\varphi_n \in \Gamma^*$ , so  $\Gamma^*$  is **complete**.  $\square$

## com.6 Construction of a Model

explanation Right now we are not concerned about  $=$ , i.e., we only want to show that a consistent set  $\Gamma$  of **sentences** not containing  $=$  is satisfiable. We first extend  $\Gamma$  to a consistent, **complete**, and saturated set  $\Gamma^*$ . In this case, the definition of a model  $\mathfrak{M}(\Gamma^*)$  is simple: We take the set of closed terms of  $\mathcal{L}'$  as the domain. We assign every **constant symbol** to itself, and make sure that more generally, for every closed term  $t$ ,  $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$ . The **predicate symbols** are assigned fol:com:mod:sec

extensions in such a way that an atomic **sentence** is true in  $\mathfrak{M}(\Gamma^*)$  iff it is in  $\Gamma^*$ . This will obviously make all the atomic **sentences** in  $\Gamma^*$  true in  $\mathfrak{M}(\Gamma^*)$ . The rest are true provided the  $\Gamma^*$  we start with is consistent, complete, and saturated.

fol:com:mod:  
defn:termmodel

**Definition com.9** (Term model). Let  $\Gamma^*$  be a **complete** and consistent, saturated set of **sentences** in a language  $\mathcal{L}$ . The *term model*  $\mathfrak{M}(\Gamma^*)$  of  $\Gamma^*$  is the **structure** defined as follows:

1. The **domain**  $|\mathfrak{M}(\Gamma^*)|$  is the set of all closed terms of  $\mathcal{L}$ .
2. The interpretation of a **constant symbol**  $c$  is  $c$  itself:  $c^{\mathfrak{M}(\Gamma^*)} = c$ .
3. The **function symbol**  $f$  is assigned the function which, given as arguments the closed terms  $t_1, \dots, t_n$ , has as value the closed term  $f(t_1, \dots, t_n)$ :

$$f^{\mathfrak{M}(\Gamma^*)}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$

4. If  $R$  is an  $n$ -place **predicate symbol**, then

$$\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)} \text{ iff } R(t_1, \dots, t_n) \in \Gamma^*.$$

A **structure**  $\mathfrak{M}$  may make an existentially quantified **sentence**  $\exists x \varphi(x)$  true without there being an instance  $\varphi(t)$  that it makes true. A **structure**  $\mathfrak{M}$  may make all instances  $\varphi(t)$  of a universally quantified **sentence**  $\forall x \varphi(x)$  true, without making  $\forall x \varphi(x)$  true. This is because in general not every **element** of  $|\mathfrak{M}|$  is the value of a closed term ( $\mathfrak{M}$  may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model  $\mathfrak{M}(\Gamma^*)$  this wouldn't be necessary—because it is covered. This is the content of the next result.

explanation

fol:com:mod:  
prop:quant-termmodel

**Proposition com.10.** Let  $\mathfrak{M}(\Gamma^*)$  be the term model of *Definition com.9*.

1.  $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$  iff  $\mathfrak{M} \models \varphi(t)$  for at least one term  $t$ .
2.  $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$  iff  $\mathfrak{M} \models \varphi(t)$  for all terms  $t$ .

*Proof.* 1. By ??,  $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$  iff for at least one variable assignment  $s$ ,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . As  $|\mathfrak{M}(\Gamma^*)|$  consists of the closed terms of  $\mathcal{L}$ , this is the case iff there is at least one closed term  $t$  such that  $s(x) = t$  and  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$  iff  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ , where  $s(x) = t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$  iff  $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ , since  $\varphi(t)$  is a sentence.

2. By ??,  $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$  iff for every variable assignment  $s$ ,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ . Recall that  $|\mathfrak{M}(\Gamma^*)|$  consists of the closed terms of  $\mathcal{L}$ , so for every closed term  $t$ ,  $s(x) = t$  is such a variable assignment, and for any variable assignment,  $s(x)$  is some closed term  $t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$  iff  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ , where  $s(x) = t$ . By ??,  $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$  iff  $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ , since  $\varphi(t)$  is a sentence.

□

**Lemma com.11** (Truth Lemma). *Suppose  $\varphi$  does not contain  $=$ . Then [fol:com:mod:lem:truth](#)  
 $\mathfrak{M}(I^*) \models \varphi$  iff  $\varphi \in I^*$ .*

*Proof.* We prove both directions simultaneously, and by induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ :  $\mathfrak{M}(I^*) \not\models \perp$  by definition of satisfaction. On the other hand,  $\perp \notin I^*$  since  $I^*$  is consistent.
2.  $\varphi \equiv \top$ :  $\mathfrak{M}(I^*) \models \top$  by definition of satisfaction. On the other hand,  $\top \in I^*$  since  $I^*$  is consistent and **complete**, and  $I^* \vdash \top$ .
3.  $\varphi \equiv R(t_1, \dots, t_n)$ :  $\mathfrak{M}(I^*) \models R(t_1, \dots, t_n)$  iff  $\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(I^*)}$  (by the definition of satisfaction) iff  $R(t_1, \dots, t_n) \in I^*$  (by the construction of  $\mathfrak{M}(I^*)$ ).
4.  $\varphi \equiv \neg\psi$ :  $\mathfrak{M}(I^*) \models \varphi$  iff  $\mathfrak{M}(I^*) \not\models \psi$  (by definition of satisfaction). By induction hypothesis,  $\mathfrak{M}(I^*) \not\models \psi$  iff  $\psi \notin I^*$ . Since  $I^*$  is consistent and **complete**,  $\psi \notin I^*$  iff  $\neg\psi \in I^*$ .
5.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}(I^*) \models \varphi$  iff we have both  $\mathfrak{M}(I^*) \models \psi$  and  $\mathfrak{M}(I^*) \models \chi$  (by definition of satisfaction) iff both  $\psi \in I^*$  and  $\chi \in I^*$  (by the induction hypothesis). By [Proposition com.2\(2\)](#), this is the case iff  $(\psi \wedge \chi) \in I^*$ .
6.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}(I^*) \models \varphi$  iff at  $\mathfrak{M}(I^*) \models \psi$  or  $\mathfrak{M}(I^*) \models \chi$  (by definition of satisfaction) iff  $\psi \in I^*$  or  $\chi \in I^*$  (by induction hypothesis). This is the case iff  $(\psi \vee \chi) \in I^*$  (by [Proposition com.2\(3\)](#)).
7.  $\varphi \equiv \psi \rightarrow \chi$ :  $\mathfrak{M}(I^*) \models \varphi$  iff  $\mathfrak{M}(I^*) \not\models \psi$  or  $\mathfrak{M}(I^*) \models \chi$  (by definition of satisfaction) iff  $\psi \notin I^*$  or  $\chi \in I^*$  (by induction hypothesis). This is the case iff  $(\psi \rightarrow \chi) \in I^*$  (by [Proposition com.2\(4\)](#)).
8.  $\varphi \equiv \forall x \psi(x)$ :  $\mathfrak{M}(I^*) \models \varphi$  iff  $\mathfrak{M}(I^*) \models \psi(t)$  for all terms  $t$  ([Proposition com.10](#)). By induction hypothesis, this is the case iff  $\psi(t) \in I^*$  for all terms  $t$ , by [Proposition com.7](#), this in turn is the case iff  $\forall x \varphi(x) \in I^*$ .
9.  $\varphi \equiv \exists x \psi(x)$ :  $\mathfrak{M}(I^*) \models \varphi$  iff  $\mathfrak{M}(I^*) \models \psi(t)$  for at least one term  $t$  ([Proposition com.10](#)). By induction hypothesis, this is the case iff  $\psi(t) \in I^*$  for at least one term  $t$ . By [Proposition com.7](#), this in turn is the case iff  $\exists x \varphi(x) \in I^*$ .

□

## com.7 Identity

[explanation](#) The construction of the term model given in the preceding section is enough to establish completeness for first-order logic for sets  $I$  that do not contain  $=$ . [fol:com:ide:sec](#)  
The term model satisfies every  $\varphi \in I^*$  which does not contain  $=$  (and hence all  $\varphi \in I$ ). It does not work, however, if  $=$  is present. The reason is that  $I^*$  then may contain a **sentence**  $t = t'$ , but in the term model the value of any

term is that term itself. Hence, if  $t$  and  $t'$  are different terms, their values in the term model—i.e.,  $t$  and  $t'$ , respectively—are different, and so  $t = t'$  is false. We can fix this, however, using a construction known as “factoring.”

**Definition com.12.** Let  $\Gamma^*$  be a consistent and **complete** set of sentences in  $\mathcal{L}$ . We define the relation  $\approx$  on the set of closed terms of  $\mathcal{L}$  by

$$t \approx t' \quad \text{iff} \quad t = t' \in \Gamma^*$$

*fol:com:ide:  
prop:approx-equiv*

**Proposition com.13.** *The relation  $\approx$  has the following properties:*

1.  $\approx$  is reflexive.
2.  $\approx$  is symmetric.
3.  $\approx$  is transitive.
4. If  $t \approx t'$ ,  $f$  is a **function symbol**, and  $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$  are terms, then

$$f(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \approx f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n).$$

5. If  $t \approx t'$ ,  $R$  is a **predicate symbol**, and  $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$  are terms, then

$$R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \in \Gamma^* \quad \text{iff} \\ R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n) \in \Gamma^*.$$

*Proof.* Since  $\Gamma^*$  is consistent and **complete**,  $t = t' \in \Gamma^*$  iff  $\Gamma^* \vdash t = t'$ . Thus it is enough to show the following:

1.  $\Gamma^* \vdash t = t$  for all terms  $t$ .
2. If  $\Gamma^* \vdash t = t'$  then  $\Gamma^* \vdash t' = t$ .
3. If  $\Gamma^* \vdash t = t'$  and  $\Gamma^* \vdash t' = t''$ , then  $\Gamma^* \vdash t = t''$ .
4. If  $\Gamma^* \vdash t = t'$ , then

$$\Gamma^* \vdash f(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) = f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$$

for every  $n$ -place **function symbol**  $f$  and terms  $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ .

5. If  $\Gamma^* \vdash t = t'$  and  $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n)$ , then  $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$  for every  $n$ -place **predicate symbol**  $R$  and terms  $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ .

□

**Problem com.2.** Complete the proof of **Proposition com.13**.

**Definition com.14.** Suppose  $\Gamma^*$  is a consistent and **complete** set in a language  $\mathcal{L}$ ,  $t$  is a term, and  $\approx$  as in the previous definition. Then:

$$[t]_{\approx} = \{t' : t' \in \text{Trm}(\mathcal{L}), t \approx t'\}$$

and  $\text{Trm}(\mathcal{L})/\approx = \{[t]_{\approx} : t \in \text{Trm}(\mathcal{L})\}$ .

**Definition com.15.** Let  $\mathfrak{M} = \mathfrak{M}(\Gamma^*)$  be the term model for  $\Gamma^*$ . Then  $\mathfrak{M}/\approx$  is the following **structure**:

fol:com:ide:  
defn:term-model-factor

1.  $|\mathfrak{M}/\approx| = \text{Trm}(\mathcal{L})/\approx$ .
2.  $c^{\mathfrak{M}/\approx} = [c]_{\approx}$
3.  $f^{\mathfrak{M}/\approx}([t_1]_{\approx}, \dots, [t_n]_{\approx}) = [f(t_1, \dots, t_n)]_{\approx}$
4.  $\langle [t_1]_{\approx}, \dots, [t_n]_{\approx} \rangle \in R^{\mathfrak{M}/\approx}$  iff  $\mathfrak{M} \models R(t_1, \dots, t_n)$ .

explanation

Note that we have defined  $f^{\mathfrak{M}/\approx}$  and  $R^{\mathfrak{M}/\approx}$  for elements of  $\text{Trm}(\mathcal{L})/\approx$  by referring to them as  $[t]_{\approx}$ , i.e., via *representatives*  $t \in [t]_{\approx}$ . We have to make sure that these definitions do not depend on the choice of these representatives, i.e., that for some other choices  $t'$  which determine the same equivalence classes ( $[t]_{\approx} = [t']_{\approx}$ ), the definitions yield the same result. For instance, if  $R$  is a one-place **predicate symbol**, the last clause of the definition says that  $[t]_{\approx} \in R^{\mathfrak{M}/\approx}$  iff  $\mathfrak{M} \models R(t)$ . If for some other term  $t'$  with  $t \approx t'$ ,  $\mathfrak{M} \not\models R(t')$ , then the definition would require  $[t']_{\approx} \notin R^{\mathfrak{M}/\approx}$ . If  $t \approx t'$ , then  $[t]_{\approx} = [t']_{\approx}$ , but we can't have both  $[t]_{\approx} \in R^{\mathfrak{M}/\approx}$  and  $[t]_{\approx} \notin R^{\mathfrak{M}/\approx}$ . However, [Proposition com.13](#) guarantees that this cannot happen.

**Proposition com.16.**  $\mathfrak{M}/\approx$  is well defined, i.e., if  $t_1, \dots, t_n, t'_1, \dots, t'_n$  are terms, and  $t_i \approx t'_i$  then

1.  $[f(t_1, \dots, t_n)]_{\approx} = [f(t'_1, \dots, t'_n)]_{\approx}$ , i.e.,  
 $f(t_1, \dots, t_n) \approx f(t'_1, \dots, t'_n)$

and

2.  $\mathfrak{M} \models R(t_1, \dots, t_n)$  iff  $\mathfrak{M} \models R(t'_1, \dots, t'_n)$ , i.e.,  
 $R(t_1, \dots, t_n) \in \Gamma^*$  iff  $R(t'_1, \dots, t'_n) \in \Gamma^*$ .

*Proof.* Follows from [Proposition com.13](#) by induction on  $n$ . □

**Lemma com.17.**  $\mathfrak{M}/\approx \models \varphi$  iff  $\varphi \in \Gamma^*$  for all sentences  $\varphi$ .

fol:com:ide:  
lem:truth

*Proof.* By induction on  $\varphi$ , just as in the proof of [Lemma com.11](#). The only case that needs additional attention is when  $\varphi \equiv t = t'$ .

$$\begin{aligned} \mathfrak{M}/\approx \models t = t' &\text{ iff } [t]_{\approx} = [t']_{\approx} \text{ (by definition of } \mathfrak{M}/\approx) \\ &\text{ iff } t \approx t' \text{ (by definition of } [t]_{\approx}) \\ &\text{ iff } t = t' \in \Gamma^* \text{ (by definition of } \approx). \end{aligned}$$

□

Note that while  $\mathfrak{M}(\Gamma^*)$  is always **enumerable** and infinite,  $\mathfrak{M}/\approx$  may be **finite**, since it may turn out that there are only finitely many classes  $[t]_{\approx}$ . This is to be expected, since  $\Gamma$  may contain **sentences** which require any **structure** in which they are true to be finite. For instance,  $\forall x \forall y x = y$  is a consistent **sentence**, but is satisfied only in **structures** with a **domain** that contains exactly one **element**. digression

## com.8 The Completeness Theorem

fol:com:cth:sec Let's combine our results: we arrive at the completeness theorem. explanation

fol:com:cth:thm:completeness **Theorem com.18** (Completeness Theorem). *Let  $\Gamma$  be a set of **sentences**. If  $\Gamma$  is consistent, it is satisfiable.*

*Proof.* Suppose  $\Gamma$  is consistent. By [Lemma com.6](#), there is a saturated consistent set  $\Gamma' \supseteq \Gamma$ . By [Lemma com.8](#), there is a  $\Gamma^* \supseteq \Gamma'$  which is consistent and **complete**. Since  $\Gamma' \subseteq \Gamma^*$ , for each **sentence**  $\varphi$ ,  $\Gamma^*$  contains a **sentence** of the form  $\exists x \varphi \rightarrow \varphi(c)$  and so  $\Gamma^*$  is saturated. If  $\Gamma$  does not contain  $=$ , then by [Lemma com.11](#),  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\varphi \in \Gamma^*$ . From this it follows in particular that for all  $\varphi \in \Gamma$ ,  $\mathfrak{M}(\Gamma^*) \models \varphi$ , so  $\Gamma$  is satisfiable. If  $\Gamma$  does contain  $=$ , then by [Lemma com.17](#),  $\mathfrak{M}/\approx \models \varphi$  iff  $\varphi \in \Gamma^*$  for all **sentences**  $\varphi$ . In particular,  $\mathfrak{M}/\approx \models \varphi$  for all  $\varphi \in \Gamma$ , so  $\Gamma$  is satisfiable.  $\square$

fol:com:cth:cor:completeness **Corollary com.19** (Completeness Theorem, Second Version). *For all  $\Gamma$  and  $\varphi$  **sentences**: if  $\Gamma \models \varphi$  then  $\Gamma \vdash \varphi$ .*

*Proof.* Note that the  $\Gamma$ 's in [Corollary com.19](#) and [Theorem com.18](#) are universally quantified. To make sure we do not confuse ourselves, let us restate [Theorem com.18](#) using a different variable: for any set of **sentences**  $\Delta$ , if  $\Delta$  is consistent, it is satisfiable. By contraposition, if  $\Delta$  is not satisfiable, then  $\Delta$  is inconsistent. We will use this to prove the corollary.

Suppose that  $\Gamma \models \varphi$ . Then  $\Gamma \cup \{\neg\varphi\}$  is unsatisfiable by **??**. Taking  $\Gamma \cup \{\neg\varphi\}$  as our  $\Delta$ , the previous version of [Theorem com.18](#) gives us that  $\Gamma \cup \{\neg\varphi\}$  is inconsistent. By **????????????????**,  $\Gamma \vdash \varphi$ .  $\square$

**Problem com.3.** Use [Corollary com.19](#) to prove [Theorem com.18](#), thus showing that the two formulations of the completeness theorem are equivalent.

**Problem com.4.** In order for a **derivation** system to be complete, its rules must be strong enough to prove every unsatisfiable set inconsistent. Which of the rules of **derivation** were necessary to prove completeness? Are any of these rules not used anywhere in the proof? In order to answer these questions, make a list or diagram that shows which of the rules of **derivation** were used in which results that lead up to the proof of [Theorem com.18](#). Be sure to note any tacit uses of rules in these proofs.

## com.9 The Compactness Theorem

One important consequence of the completeness theorem is the compactness theorem. The compactness theorem states that if each *finite* subset of a set of **sentences** is satisfiable, the entire set is satisfiable—even if the set itself is infinite. This is far from obvious. There is nothing that seems to rule out, at first glance at least, the possibility of there being infinite sets of **sentences** which are contradictory, but the contradiction only arises, so to speak, from the infinite number. The compactness theorem says that such a scenario can be ruled out: there are no unsatisfiable infinite sets of **sentences** each finite subset of which is satisfiable. Like the completeness theorem, it has a version related to entailment: if an infinite set of **sentences** entails something, already a finite subset does.

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sec

**Definition com.20.** A set  $\Gamma$  of **formulas** is *finitely satisfiable* if and only if every finite  $\Gamma_0 \subseteq \Gamma$  is satisfiable.

**Theorem com.21** (Compactness Theorem). *The following hold for any sentences  $\Gamma$  and  $\varphi$ :*

fol:com:com:  
thm:compactness

1.  $\Gamma \models \varphi$  iff there is a finite  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \models \varphi$ .
2.  $\Gamma$  is satisfiable if and only if it is finitely satisfiable.

*Proof.* We prove (2). If  $\Gamma$  is satisfiable, then there is a **structure**  $\mathfrak{M}$  such that  $\mathfrak{M} \models \varphi$  for all  $\varphi \in \Gamma$ . Of course, this  $\mathfrak{M}$  also satisfies every finite subset of  $\Gamma$ , so  $\Gamma$  is finitely satisfiable.

Now suppose that  $\Gamma$  is finitely satisfiable. Then every finite subset  $\Gamma_0 \subseteq \Gamma$  is satisfiable. By soundness (????????????????), every finite subset is consistent. Then  $\Gamma$  itself must be consistent by ??????????????????. By completeness (Theorem com.18), since  $\Gamma$  is consistent, it is satisfiable.  $\square$

**Problem com.5.** Prove (1) of Theorem com.21.

**Example com.22.** In every model  $\mathfrak{M}$  of a theory  $\Gamma$ , each term  $t$  of course picks out an **element** of  $|\mathfrak{M}|$ . Can we guarantee that it is also true that every **element** of  $|\mathfrak{M}|$  is picked out by some term or other? In other words, are there theories  $\Gamma$  all models of which are covered? The compactness theorem shows that this is not the case if  $\Gamma$  has infinite models. Here's how to see this: Let  $\mathfrak{M}$  be an infinite model of  $\Gamma$ , and let  $c$  be a **constant symbol** not in the language of  $\Gamma$ . Let  $\Delta$  be the set of all sentences  $c \neq t$  for  $t$  a term in the language  $\mathcal{L}$  of  $\Gamma$ , i.e.,

$$\Delta = \{c \neq t : t \in \text{Trm}(\mathcal{L})\}.$$

A finite subset of  $\Gamma \cup \Delta$  can be written as  $\Gamma' \cup \Delta'$ , with  $\Gamma' \subseteq \Gamma$  and  $\Delta' \subseteq \Delta$ . Since  $\Delta'$  is finite, it can contain only finitely many terms. Let  $a \in |\mathfrak{M}|$  be an **element** of  $|\mathfrak{M}|$  not picked out by any of them, and let  $\mathfrak{M}'$  be the **structure** that is just like  $\mathfrak{M}$ , but also  $c^{\mathfrak{M}'} = a$ . Since  $a \neq \text{Val}^{\mathfrak{M}}(t)$  for all  $t$  occurring in  $\Delta'$ ,  $\mathfrak{M}' \models \Delta'$ . Since  $\mathfrak{M} \models \Gamma$ ,  $\Gamma' \subseteq \Gamma$ , and  $c$  does not occur in  $\Gamma$ , also  $\mathfrak{M}' \models \Gamma'$ .

Together,  $\mathfrak{M}' \models \Gamma' \cup \Delta'$  for every finite subset  $\Gamma' \cup \Delta'$  of  $\Gamma \cup \Delta$ . So every finite subset of  $\Gamma \cup \Delta$  is satisfiable. By compactness,  $\Gamma \cup \Delta$  itself is satisfiable. So there are models  $\mathfrak{M} \models \Gamma \cup \Delta$ . Every such  $\mathfrak{M}$  is a model of  $\Gamma$ , but is not covered, since  $\text{Val}^{\mathfrak{M}}(c) \neq \text{Val}^{\mathfrak{M}}(t)$  for all terms  $t$  of  $\mathcal{L}$ .

**Example com.23.** Consider a language  $\mathcal{L}$  containing the **predicate symbol**  $<$ , **constant symbols**  $0, 1$ , and **function symbols**  $+, \times, -, \div$ . Let  $\Gamma$  be the set of all **sentences** in this language true in  $\mathfrak{Q}$  with domain  $\mathbb{Q}$  and the obvious interpretations.  $\Gamma$  is the set of all **sentences** of  $\mathcal{L}$  true about the rational numbers. Of course, in  $\mathbb{Q}$  (and even in  $\mathbb{R}$ ), there are no numbers which are greater than  $0$  but less than  $1/k$  for all  $k \in \mathbb{Z}^+$ . Such a number, if it existed, would be an *infinitesimal*: non-zero, but infinitely small. The compactness theorem shows that there are models of  $\Gamma$  in which infinitesimals exist: Let  $\Delta$  be  $\{0 < c\} \cup \{c < (1 \div \bar{k}) : k \in \mathbb{Z}^+\}$  (where  $\bar{k} = (1 + (1 + \dots + (1 + 1) \dots))$  with  $k$   $1$ 's). For any finite subset  $\Delta_0$  of  $\Delta$  there is a  $K$  such that all the **sentences**  $c < \bar{k}$  in  $\Delta_0$  have  $k < K$ . If we expand  $\mathfrak{Q}$  to  $\mathfrak{Q}'$  with  $c^{\mathfrak{Q}'} = 1/K$  we have that  $\mathfrak{Q}' \models \Gamma \cup \Delta_0$ , and so  $\Gamma \cup \Delta$  is finitely satisfiable (Exercise: prove this in detail). By compactness,  $\Gamma \cup \Delta$  is satisfiable. Any model  $\mathfrak{S}$  of  $\Gamma \cup \Delta$  contains an infinitesimal, namely  $c^{\mathfrak{S}}$ .

**Problem com.6.** In the standard model of arithmetic  $\mathfrak{N}$ , there is no **element**  $k \in |\mathfrak{N}|$  which satisfies every formula  $\bar{n} < x$  (where  $\bar{n}$  is  $0' \dots'$  with  $n$   $1$ 's). Use the compactness theorem to show that the set of **sentences** in the language of arithmetic which are true in the standard model of arithmetic  $\mathfrak{N}$  are also true in a **structure**  $\mathfrak{N}'$  that contains an **element** which *does* satisfy every formula  $\bar{n} < x$ .

**Example com.24.** We know that first-order logic with **identity predicate** can express that the size of the **domain** must have some minimal size: The sentence  $\varphi_{\geq n}$  (which says “there are at least  $n$  distinct objects”) is true only in structures where  $|\mathfrak{M}|$  has at least  $n$  objects. So if we take

$$\Delta = \{\varphi_{\geq n} : n \geq 1\}$$

then any model of  $\Delta$  must be infinite. Thus, we can guarantee that a theory only has infinite models by adding  $\Delta$  to it: the models of  $\Gamma \cup \Delta$  are all and only the infinite models of  $\Gamma$ .

So first-order logic can express infinitude. The compactness theorem shows that it cannot express finitude, however. For suppose some set of sentences  $A$  were satisfied in all and only finite **structures**. Then  $\Delta \cup A$  is finitely satisfiable. Why? Suppose  $\Delta' \cup A' \subseteq \Delta \cup A$  is finite with  $\Delta' \subseteq \Delta$  and  $A' \subseteq A$ . Let  $n$  be the largest number such that  $\varphi_{\geq n} \in \Delta'$ .  $A$ , being satisfied in all finite structures, has a model  $\mathfrak{M}$  with finitely many but  $\geq n$  **elements**. But then  $\mathfrak{M} \models \Delta' \cup A'$ . By compactness,  $\Delta \cup A$  has an infinite model, contradicting the assumption that  $A$  is satisfied only in finite **structures**.

## com.10 A Direct Proof of the Compactness Theorem

We can prove the Compactness Theorem directly, without appealing to the Completeness Theorem, using the same ideas as in the proof of the completeness theorem. In the proof of the Completeness Theorem we started with a consistent set  $\Gamma$  of sentences, expanded it to a consistent, saturated, and complete set  $\Gamma^*$  of sentences, and then showed that in the term model  $\mathfrak{M}(\Gamma^*)$  constructed from  $\Gamma^*$ , all sentences of  $\Gamma$  are true, so  $\Gamma$  is satisfiable.

We can use the same method to show that a finitely satisfiable set of sentences is satisfiable. We just have to prove the corresponding versions of the results leading to the truth lemma where we replace “consistent” with “finitely satisfiable.”

**Proposition com.25.** *Suppose  $\Gamma$  is complete and finitely satisfiable. Then:* fol:com:cpd:sec  
prop:fsat-ccs

1.  $(\varphi \wedge \psi) \in \Gamma$  iff both  $\varphi \in \Gamma$  and  $\psi \in \Gamma$ .
2.  $(\varphi \vee \psi) \in \Gamma$  iff either  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ .
3.  $(\varphi \rightarrow \psi) \in \Gamma$  iff either  $\varphi \notin \Gamma$  or  $\psi \in \Gamma$ .

**Problem com.7.** Prove Proposition com.25. Avoid the use of  $\vdash$ .

**Lemma com.26.** *Every finitely satisfiable set  $\Gamma$  can be extended to a saturated finitely satisfiable set  $\Gamma'$ .* fol:com:cpd:lem:fsat-henkin

**Problem com.8.** Prove Lemma com.26. (Hint: The crucial step is to show that if  $\Gamma_n$  is finitely satisfiable, so is  $\Gamma_n \cup \{\theta_n\}$ , without any appeal to derivations or consistency.)

**Proposition com.27.** *Suppose  $\Gamma$  is complete, finitely satisfiable, and saturated.* fol:com:cpd:prop:fsat-instances

1.  $\exists x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for at least one closed term  $t$ .
2.  $\forall x \varphi(x) \in \Gamma$  iff  $\varphi(t) \in \Gamma$  for all closed terms  $t$ .

**Problem com.9.** Prove Proposition com.27.

**Lemma com.28.** *Every finitely satisfiable set  $\Gamma$  can be extended to a complete and finitely satisfiable set  $\Gamma^*$ .* fol:com:cpd:lem:fsat-lindenbaum

**Problem com.10.** Prove Lemma com.28. (Hint: the crucial step is to show that if  $\Gamma_n$  is finitely satisfiable, then either  $\Gamma_n \cup \{\varphi_n\}$  or  $\Gamma_n \cup \{\neg\varphi_n\}$  is finitely satisfiable.)

**Theorem com.29** (Compactness).  *$\Gamma$  is satisfiable if and only if it is finitely satisfiable.* fol:com:cpd:thm:compactness-direct

*Proof.* If  $\Gamma$  is satisfiable, then there is a **structure**  $\mathfrak{M}$  such that  $\mathfrak{M} \models \varphi$  for all  $\varphi \in \Gamma$ . Of course, this  $\mathfrak{M}$  also satisfies every finite subset of  $\Gamma$ , so  $\Gamma$  is finitely satisfiable.

Now suppose that  $\Gamma$  is finitely satisfiable. By [Lemma com.26](#), there is a finitely satisfiable, saturated set  $\Gamma' \supseteq \Gamma$ . By [Lemma com.28](#),  $\Gamma'$  can be extended to a **complete** and finitely satisfiable set  $\Gamma^*$ , and  $\Gamma^*$  is still saturated. Construct the term model  $\mathfrak{M}(\Gamma^*)$  as in [Definition com.9](#). Note that [Proposition com.10](#) did not rely on the fact that  $\Gamma^*$  is consistent (or **complete** or saturated, for that matter), but just on the fact that  $\mathfrak{M}(\Gamma^*)$  is covered. The proof of the Truth Lemma ([Lemma com.11](#)) goes through if we replace references to [Proposition com.2](#) and [Proposition com.7](#) by references to [Proposition com.25](#) and [Proposition com.27](#)  $\square$

**Problem com.11.** Write out the complete proof of the Truth Lemma ([Lemma com.11](#)) in the version required for the proof of [Theorem com.29](#).

## com.11 The Löwenheim-Skolem Theorem

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The Löwenheim-Skolem Theorem says that if a theory has an infinite model, then it also has a model that is at most **denumerable**. An immediate consequence of this fact is that first-order logic cannot express that the size of a **structure** is **non-enumerable**: any **sentence** or set of **sentences** satisfied in all **non-enumerable structures** is also satisfied in some **enumerable** structure.

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**Theorem com.30.** *If  $\Gamma$  is consistent then it has an **enumerable** model, i.e., it is satisfiable in a **structure** whose domain is either finite or **denumerable**.*

*Proof.* If  $\Gamma$  is consistent, the **structure**  $\mathfrak{M}$  delivered by the proof of the completeness theorem has a domain  $|\mathfrak{M}|$  that is no larger than the set of the terms of the language  $\mathcal{L}$ . So  $\mathfrak{M}$  is at most **denumerable**.  $\square$

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[noidentity-ls](#)

**Theorem com.31.** *If  $\Gamma$  is consistent set of **sentences** in the language of first-order logic without identity, then it has a **denumerable** model, i.e., it is satisfiable in a **structure** whose domain is infinite and **enumerable**.*

*Proof.* If  $\Gamma$  is consistent and contains no sentences in which identity appears, then the **structure**  $\mathfrak{M}$  delivered by the proof of the completeness theorem has a domain  $|\mathfrak{M}|$  identical to the set of terms of the language  $\mathcal{L}'$ . So  $\mathfrak{M}$  is **denumerable**, since  $\text{Trm}(\mathcal{L}')$  is.  $\square$

**Example com.32** (Skolem's Paradox). Zermelo-Fraenkel set theory **ZFC** is a very powerful framework in which practically all mathematical statements can be expressed, including facts about the sizes of sets. So for instance, **ZFC** can prove that the set  $\mathbb{R}$  of real numbers is **non-enumerable**, it can prove Cantor's Theorem that the power set of any set is larger than the set itself, etc. If **ZFC** is consistent, its models are all infinite, and moreover, they all contain **elements** about which the theory says that they are **non-enumerable**, such as

the element that makes true the theorem of **ZFC** that the power set of the natural numbers exists. By the Löwenheim-Skolem Theorem, **ZFC** also has **enumerable** models—models that contain “**non-enumerable**” sets but which themselves are **enumerable**.

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# Bibliography