

## com.1 The Completeness Theorem

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sec

Let's combine our results: we arrive at the completeness theorem.

explanation

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thm:completeness

**Theorem com.1** (Completeness Theorem). *Let  $\Gamma$  be a set of sentences. If  $\Gamma$  is consistent, it is satisfiable.*

*Proof.* Suppose  $\Gamma$  is consistent. By ??, there is a saturated consistent set  $\Gamma' \supseteq \Gamma$ . By ??, there is a  $\Gamma^* \supseteq \Gamma'$  which is consistent and complete. Since  $\Gamma' \subseteq \Gamma^*$ , for each sentence  $\varphi$ ,  $\Gamma^*$  contains a sentence of the form  $\exists x \varphi \rightarrow \varphi(c)$  and so  $\Gamma^*$  is saturated. If  $\Gamma$  does not contain  $=$ , then by ??,  $\mathfrak{M}(\Gamma^*) \models \varphi$  iff  $\varphi \in \Gamma^*$ . From this it follows in particular that for all  $\varphi \in \Gamma$ ,  $\mathfrak{M}(\Gamma^*) \models \varphi$ , so  $\Gamma$  is satisfiable. If  $\Gamma$  does contain  $=$ , then by ??,  $\mathfrak{M}/\approx \models \varphi$  iff  $\varphi \in \Gamma^*$  for all sentences  $\varphi$ . In particular,  $\mathfrak{M}/\approx \models \varphi$  for all  $\varphi \in \Gamma$ , so  $\Gamma$  is satisfiable.  $\square$

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cor:completeness

**Corollary com.2** (Completeness Theorem, Second Version). *For all  $\Gamma$  and  $\varphi$  sentences: if  $\Gamma \models \varphi$  then  $\Gamma \vdash \varphi$ .*

*Proof.* Note that the  $\Gamma$ 's in Corollary com.2 and Theorem com.1 are universally quantified. To make sure we do not confuse ourselves, let us restate Theorem com.1 using a different variable: for any set of sentences  $\Delta$ , if  $\Delta$  is consistent, it is satisfiable. By contraposition, if  $\Delta$  is not satisfiable, then  $\Delta$  is inconsistent. We will use this to prove the corollary.

Suppose that  $\Gamma \models \varphi$ . Then  $\Gamma \cup \{\neg\varphi\}$  is unsatisfiable by ??. Taking  $\Gamma \cup \{\neg\varphi\}$  as our  $\Delta$ , the previous version of Theorem com.1 gives us that  $\Gamma \cup \{\neg\varphi\}$  is inconsistent. By ??????????????,  $\Gamma \vdash \varphi$ .  $\square$

**Problem com.1.** Use Corollary com.2 to prove Theorem com.1, thus showing that the two formulations of the completeness theorem are equivalent.

**Problem com.2.** In order for a derivation system to be complete, its rules must be strong enough to prove every unsatisfiable set inconsistent. Which of the rules of derivation were necessary to prove completeness? Are any of these rules not used anywhere in the proof? In order to answer these questions, make a list or diagram that shows which of the rules of derivation were used in which results that lead up to the proof of Theorem com.1. Be sure to note any tacit uses of rules in these proofs.

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## Bibliography