com.1 Complete Consistent Sets of Sentences

Definition com.1 (Complete set). A set $\Gamma$ of sentences is complete iff for any sentence $\varphi$, either $\varphi \in \Gamma$ or $\neg \varphi \in \Gamma$.

Complete sets of sentences leave no questions unanswered. For any sentence $\varphi$, $\Gamma$ “says” if $\varphi$ is true or false. The importance of complete sets extends beyond the proof of the completeness theorem. A theory which is complete and axiomatizable, for instance, is always decidable.

Complete consistent sets are important in the completeness proof since we can guarantee that every consistent set of sentences $\Gamma$ is contained in a complete consistent set $\Gamma^*$. A complete consistent set contains, for each sentence $\varphi$, either $\varphi$ or its negation $\neg \varphi$, but not both. This is true in particular for atomic sentences, so from a complete consistent set in a language suitably expanded by constant symbols, we can construct a structure where the interpretation of predicate symbols is defined according to which atomic sentences are in $\Gamma^*$. This structure can then be shown to make all sentences in $\Gamma^*$ (and hence also all those in $\Gamma$) true. The proof of this latter fact requires that $\neg \varphi \in \Gamma^*$ iff $\varphi \notin \Gamma^*$, $(\varphi \lor \psi) \in \Gamma^*$ iff $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$, etc.

In what follows, we will often tacitly use the properties of reflexivity, monotonicity, and transitivity of $\vdash$ (see ?????????????).

Proposition com.2. Suppose $\Gamma$ is complete and consistent. Then:

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.
2. $\varphi \land \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.
3. $\varphi \lor \psi \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.
4. $\varphi \rightarrow \psi \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Proof. Let us suppose for all of the following that $\Gamma$ is complete and consistent.

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

Suppose that $\Gamma \vdash \varphi$. Suppose to the contrary that $\varphi \notin \Gamma$. Since $\Gamma$ is complete, $\neg \varphi \in \Gamma$. By ?????????????, $\Gamma$ is inconsistent. This contradicts the assumption that $\Gamma$ is consistent. Hence, it cannot be the case that $\varphi \notin \Gamma$, so $\varphi \in \Gamma$.

2. $\varphi \land \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$:

For the forward direction, suppose $\varphi \land \psi \in \Gamma$. Then by ?????????????, item (1), $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$. By (1), $\varphi \in \Gamma$ and $\psi \in \Gamma$, as required.

For the reverse direction, let $\varphi \in \Gamma$ and $\psi \in \Gamma$. By ?????????????, item (2), $\Gamma \vdash \varphi \land \psi$. By (1), $\varphi \land \psi \in \Gamma$.  

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3. First we show that if $\varphi \lor \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose $\varphi \lor \psi \in \Gamma$ but $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since $\Gamma$ is complete, $\neg \varphi \in \Gamma$ and $\neg \psi \in \Gamma$. By ??????????????, item (1), $\Gamma$ is inconsistent, a contradiction. Hence, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

For the reverse direction, suppose that $\varphi \in \Gamma$ or $\psi \in \Gamma$. By ??????????????, item (2), $\Gamma \vdash \varphi \lor \psi$. By (1), $\varphi \lor \psi \in \Gamma$, as required.

4. For the forward direction, suppose $\varphi \rightarrow \psi \in \Gamma$, and suppose to the contrary that $\varphi \in \Gamma$ and $\psi \notin \Gamma$. On these assumptions, $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$. By ??????????????, item (1), $\Gamma \vdash \psi$. But then by (1), $\psi \in \Gamma$, contradicting the assumption that $\psi \notin \Gamma$.

For the reverse direction, first consider the case where $\varphi \notin \Gamma$. Since $\Gamma$ is complete, $\neg \varphi \in \Gamma$. By ??????????????, item (2), $\Gamma \vdash \varphi \rightarrow \psi$. Again by (1), we get that $\varphi \rightarrow \psi \in \Gamma$, as required.

Now consider the case where $\psi \in \Gamma$. By ??????????????, item (2) again, $\Gamma \vdash \varphi \rightarrow \psi$. By (1), $\varphi \rightarrow \psi \in \Gamma$. $\square$

Problem com.1. Complete the proof of Proposition com.2.

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Bibliography