com.1 A Direct Proof of the Compactness Theorem

We can prove the Compactness Theorem directly, without appealing to the Completeness Theorem, using the same ideas as in the proof of the completeness theorem. In the proof of the Completeness Theorem we started with a consistent set Γ of sentences, expanded it to a consistent, saturated, and complete set Γ^* of sentences, and then showed that in the term model $\mathfrak{M}(\Gamma^*)$ constructed from Γ^*, all sentences of Γ are true, so Γ is satisfiable.

We can use the same method to show that a finitely satisfiable set of sentences is satisfiable. We just have to prove the corresponding versions of the results leading to the truth lemma where we replace "consistent" with "finitely satisfiable."

Proposition com.1. *Suppose Γ is complete and finitely satisfiable. Then:*

1. *$(\varphi \wedge \psi) \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.*

2. *$(\varphi \vee \psi) \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

3. *$(\varphi \to \psi) \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.*

Problem com.1. Prove Proposition com.1. Avoid the use of $\vdash$.

Lemma com.2. *Every finitely satisfiable set Γ can be extended to a saturated finitely satisfiable set Γ'.*

Problem com.2. Prove Lemma com.2. (Hint: The crucial step is to show that if Γ_n is finitely satisfiable, so is $\Gamma_n \cup \{\theta_n\}$, without any appeal to derivations or consistency.)

Proposition com.3. *Suppose Γ is complete, finitely satisfiable, and saturated.*

1. *$\exists x\, \varphi(x) \in \Gamma$ iff $\varphi(t) \in \Gamma$ for at least one closed term t.*

2. *$\forall x\, \varphi(x) \in \Gamma$ iff $\varphi(t) \in \Gamma$ for all closed terms t.*

Problem com.3. Prove Proposition com.3.

Lemma com.4. *Every finitely satisfiable set Γ can be extended to a complete and finitely satisfiable set Γ^*.*

Problem com.4. Prove Lemma com.4. (Hint: the crucial step is to show that if Γ_n is finitely satisfiable, then either $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is finitely satisfiable.)

Theorem com.5 (Compactness)**.** *Γ is satisfiable if and only if it is finitely satisfiable.*

Proof. If Γ is satisfiable, then there is a structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash \varphi$ for all $\varphi \in \Gamma$. Of course, this $\mathfrak{M}$ also satisfies every finite subset of Γ, so Γ is finitely satisfiable.

Now suppose that Γ is finitely satisfiable. By Lemma com.2, there is a finitely satisfiable, saturated set $\Gamma' \supseteq \Gamma$. By Lemma com.4, Γ' can be extended to a complete and finitely satisfiable set Γ^*, and Γ^* is still saturated. Construct the term model $\mathfrak{M}(\Gamma^*)$ as in **??**. Note that **??** did not rely on the fact that Γ^* is consistent (or complete or saturated, for that matter), but just on the fact that $\mathfrak{M}(\Gamma^*)$ is covered. The proof of the Truth Lemma (**??**) goes through if we replace references to **??** and **??** by references to Proposition com.1 and Proposition com.3 $\qquad\square$

Problem com.5. Write out the complete proof of the Truth Lemma (**??**) in the version required for the proof of Theorem com.5.

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Bibliography