byd.1 Modal Logics

Consider the following example of a conditional sentence:

If Jeremy is alone in that room, then he is drunk and naked and
dancing on the chairs.

This is an example of a conditional assertion that may be materially true
but nonetheless misleading, since it seems to suggest that there is a stronger
link between the antecedent and conclusion other than simply that either the
antecedent is false or the consequent true. That is, the wording suggests that
the claim is not only true in this particular world (where it may be trivially true,
because Jeremy is not alone in the room), but that, moreover, the conclusion
would have been true had the antecedent been true. In other words, one can
take the assertion to mean that the claim is true not just in this world, but in
any “possible” world; or that it is necessarily true, as opposed to just true in
this particular world.

Modal logic was designed to make sense of this kind of necessity. One
obtains modal propositional logic from ordinary propositional logic by adding
a box operator; which is to say, if \( \varphi \) is a formula, so is \( \square \varphi \). Intuitively, \( \square \varphi \)
asserts that \( \varphi \) is necessarily true, or true in any possible world. \( \diamond \varphi \) is usually
taken to be an abbreviation for \( \neg \square \neg \varphi \), and can be read as asserting that \( \varphi \) is
possibly true. Of course, modality can be added to predicate logic as well.

Kripke structures can be used to provide a semantics for modal logic; in
fact, Kripke first designed this semantics with modal logic in mind. Rather than
restricting to partial orders, more generally one has a set of “possible worlds,”
\( P \), and a binary “accessibility” relation \( R(x, y) \) between worlds. Intuitively,
\( R(p, q) \) asserts that the world \( q \) is compatible with \( p \); i.e., if we are “in” world \( p \),
we have to entertain the possibility that the world could have been like \( q \).

Modal logic is sometimes called an “intensional” logic, as opposed to an
“extensional” one. The intended semantics for an extensional logic, like classical
logic, will only refer to a single world, the “actual” one; while the semantics
for an “intensional” logic relies on a more elaborate ontology. In addition to
structureing necessity, one can use modality to structure other linguistic con-
structions, reinterpreting \( \square \) and \( \diamond \) according to the application. For example:

1. In provability logic, \( \square \varphi \) is read “\( \varphi \) is provable” and \( \diamond \varphi \) is read “\( \varphi \) is
consistent.”

2. In epistemic logic, one might read \( \square \varphi \) as “I know \( \varphi \)” or “I believe \( \varphi \).”

3. In temporal logic, one can read \( \square \varphi \) as “\( \varphi \) is always true” and \( \diamond \varphi \) as “\( \varphi \)
is sometimes true.”

One would like to augment logic with rules and axioms dealing with modality. For example, the system \( \textbf{S4} \) consists of the ordinary axioms and rules of
propositional logic, together with the following axioms:

\[ \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \]
\[ \Box \varphi \rightarrow \varphi \]
\[ \Box \varphi \rightarrow \Box \Box \varphi \]

as well as a rule, “from \( \varphi \) conclude \( \Box \varphi \).” \( \textbf{S5} \) adds the following axiom:

\[ \Diamond \varphi \rightarrow \Box \Diamond \varphi \]

Variations of these axioms may be suitable for different applications; for example, \( \textbf{S5} \) is usually taken to characterize the notion of logical necessity. And the nice thing is that one can usually find a semantics for which the proof system is sound and complete by restricting the accessibility relation in the Kripke structures in natural ways. For example, \( \textbf{S4} \) corresponds to the class of Kripke structures in which the accessibility relation is reflexive and transitive. \( \textbf{S5} \) corresponds to the class of Kripke structures in which the accessibility relation is universal, which is to say that every world is accessible from every other; so \( \Box \varphi \) holds if and only if \( \varphi \) holds in every world.

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\textbf{Bibliography}