axd.1 Soundness

fol:axd:sou:sec

A derivation system, such as axiomatic deduction, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivable φ is valid;
2. if φ is derivable from some others Γ, it is also a consequence of them;
3. if a set of formulas Γ is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Proposition axd.1. *If φ is an axiom, then $\mathfrak{M}, s \vDash \varphi$ for each structure $\mathfrak{M}$ and assignment s.*

Proof. We have to verify that all the axioms are valid. For instance, here is the case for **??**: suppose t is free for x in φ, and assume $\mathfrak{M}, s \vDash \forall x\, \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \vDash \varphi$, and in particular this holds when $s'(x) = \mathrm{Val}^{\mathfrak{M}}_s(t)$. By **??**, $\mathfrak{M}, s \vDash \varphi[t/x]$. This shows that $\mathfrak{M}, s \vDash (\forall x\, \varphi \to \varphi[t/x])$. □

fol:axd:sou:thm:soundness

Theorem axd.2 (Soundness). *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Proof. By induction on the length of the derivation of φ from Γ. If there are no steps justified by inferences, then all formulas in the derivation are either instances of axioms or are in Γ. By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \vDash \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \vDash \varphi$.

If the last step of the derivation of φ is justified by modus ponens, then there are formulas ψ and $\psi \to \varphi$ in the derivation, and the induction hypothesis applies to the part of the derivation ending in those formulas (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \to \varphi$. Then $\Gamma \vDash \varphi$ by **??**.

Now suppose the last step is justified by QR. Then that step has the form $\chi \to \forall x\, B(x)$ and there is a preceding step $\chi \to \psi(c)$ with c not in Γ, χ, or $\forall x\, B(x)$. By induction hypothesis, $\Gamma \vDash \chi \to \forall x\, B(x)$. By **??**, $\Gamma \cup \{\chi\} \vDash \psi(c)$.

Consider some structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \vDash \forall x\, \psi(x)$. Since $\forall x\, \psi(x)$ is a sentence, this means we have to show that for every variable assignment s, $\mathfrak{M}, s \vDash \psi(x)$ (**??**). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \vDash \theta$ for all $\theta \in \Gamma$ by **??**. Let $\mathfrak{M}'$ be like $\mathfrak{M}$ except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ, $\mathfrak{M}' \vDash \Gamma \cup \{\chi\}$ by **??**. Since $\Gamma \cup \{\chi\} \vDash \psi(c)$, $\mathfrak{M}' \vDash B(c)$. Since $\psi(c)$ is a sentence, $\mathfrak{M}, s \vDash \psi(c)$ by **??**. $\mathfrak{M}', s \vDash \psi(x)$ iff $\mathfrak{M}' \vDash \psi(c)$ by **??** (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So,

$\mathfrak{M}', s \vDash \psi(x)$. Since c does not occur in $\psi(x)$, by **??**, $\mathfrak{M}, s \vDash \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \vDash \forall x\, \psi(x)$. Thus $\Gamma \cup \{\chi\} \vDash \forall x\, \psi(x)$. By **??**, $\Gamma \vDash \chi \rightarrow \forall x\, \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x\, \psi(x) \rightarrow \chi$ is left as an exercise. □

Problem axd.1. Complete the proof of Theorem axd.2.

Corollary axd.3. *If $\vdash \varphi$, then φ is valid.* *fol:axd:sou: cor:weak-soundness*

Corollary axd.4. *If Γ is satisfiable, then it is consistent.* *fol:axd:sou: cor:consistency-soundness*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from Γ. By Theorem axd.2, any structure $\mathfrak{M}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{M} \nvDash \bot$ for every structure $\mathfrak{M}$, no $\mathfrak{M}$ can satisfy Γ, i.e., Γ is not satisfiable. □

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Bibliography