

axd.1 Soundness

fol:axd:sou: sec A **derivation** system, such as axiomatic deduction, is *sound* if it cannot **derive** explanation things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every **derivable** φ is valid;
2. if φ is **derivable** from some others Γ , it is also a consequence of them;
3. if a set of **formulas** Γ is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Proposition axd.1. *If φ is an axiom, then $\mathfrak{M}, s \models \varphi$ for each **structure** \mathfrak{M} and **assignment** s .*

Proof. We have to verify that all the axioms are valid. For instance, here is the case for **??**: suppose t is **free** for x in φ , and assume $\mathfrak{M}, s \models \forall x \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \models \varphi$, and in particular this holds when $s'(x) = \text{Val}_s^{\mathfrak{M}}(t)$. By **??**, $\mathfrak{M}, s \models \varphi[t/x]$. This shows that $\mathfrak{M}, s \models (\forall x \varphi \rightarrow \varphi[t/x])$. \square

fol:axd:sou: thm:soundness **Theorem axd.2 (Soundness).** *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Proof. By induction on the length of the **derivation** of φ from Γ . If there are no steps justified by inferences, then all **formulas** in the derivation are either instances of axioms or are in Γ . By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \models \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \models \varphi$.

If the last step of the derivation of φ is justified by modus ponens, then there are **formulas** ψ and $\psi \rightarrow \varphi$ in the **derivation**, and the induction hypothesis applies to the part of the **derivation** ending in those **formulas** (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \models \psi$ and $\Gamma \models \psi \rightarrow \varphi$. Then $\Gamma \models \varphi$ by **??**.

Now suppose the last step is justified by QR. Then that step has the form $\chi \rightarrow \forall x B(x)$ and there is a preceding step $\chi \rightarrow \psi(c)$ with c not in Γ , χ , or $\forall x B(x)$. By induction hypothesis, $\Gamma \models \chi \rightarrow \forall x B(x)$. By **??**, $\Gamma \cup \{\chi\} \models \psi(c)$.

Consider some structure \mathfrak{M} such that $\mathfrak{M} \models \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \models \forall x \psi(x)$. Since $\forall x \psi(x)$ is a **sentence**, this means we have to show that for every variable assignment s , $\mathfrak{M}, s \models \psi(x)$ (**??**). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \models \theta$ for all $\theta \in \Gamma$ by **??**. Let \mathfrak{M}' be like \mathfrak{M} except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ , $\mathfrak{M}' \models \Gamma \cup \{\chi\}$ by **??**. Since $\Gamma \cup \{\chi\} \models \psi(c)$, $\mathfrak{M}' \models B(c)$. Since $\psi(c)$ is a **sentence**, $\mathfrak{M}, s \models \psi(c)$ by **??**. $\mathfrak{M}', s \models \psi(x)$ iff $\mathfrak{M}' \models \psi(c)$ by **??** (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So,

$\mathfrak{M}, s \models \psi(x)$. Since c does not occur in $\psi(x)$, by ??, $\mathfrak{M}, s \models \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \models \forall x \psi(x)$. Thus $\Gamma \cup \{\chi\} \models \forall x \psi(x)$. By ??, $\Gamma \models \chi \rightarrow \forall x \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x \psi(x) \rightarrow \chi$ is left as an exercise. \square

Problem axd.1. Complete the proof of **Theorem axd.2**.

Corollary axd.3. *If $\vdash \varphi$, then φ is valid.*

*fol:axd:sou:
cor:weak-soundness*

Corollary axd.4. *If Γ is satisfiable, then it is consistent.*

*fol:axd:sou:
cor:consistency-soundness*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \perp$, i.e., there is a **derivation** of \perp from Γ . By **Theorem axd.2**, any **structure** \mathfrak{M} that satisfies Γ must satisfy \perp . Since $\mathfrak{M} \not\models \perp$ for every **structure** \mathfrak{M} , no \mathfrak{M} can satisfy Γ , i.e., Γ is not satisfiable. \square

Photo Credits

Bibliography