Axid.1 Rules and Derivations

Axiomatic derivations are perhaps the simplest derivation system for logic. A derivation is just a sequence of formulas. To count as a derivation, every formula in the sequence must either be an instance of an axiom, or must follow from one or more formulas that precede it in the sequence by a rule of inference. A derivation derives its last formula.

Definition axd.1 (Derivability). If $\Gamma$ is a set of formulas of $L$ then a derivation from $\Gamma$ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. $\varphi_i$ is an axiom; or
3. $\varphi_i$ follows from some $\varphi_j$ (and $\varphi_k$) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct derivation depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step $A_i$ in a derivation is a correct inference step.

Definition axd.2 (Rule of inference). A rule of inference gives a sufficient condition for what counts as a correct inference step in a derivation from $\Gamma$.

For instance, since any one-element sequence $\varphi$ with $\varphi \in \Gamma$ trivially counts as a derivation, the following might be a very simple rule of inference:

If $\varphi \in \Gamma$, then $\varphi$ is always a correct inference step in any derivation from $\Gamma$.

Similarly, if $\varphi$ is one of the axioms, then $\varphi$ by itself is a derivation, and so this is also a rule of inference:

If $\varphi$ is an axiom, then $\varphi$ is a correct inference step.

It gets more interesting if the rule of inference appeals to formulas that appear before the step considered. The following rule is called modus ponens:

If $\psi \rightarrow \varphi$ and $\psi$ occur higher up in the derivation, then $\varphi$ is a correct inference step.

If this is the only rule of inference, then our definition of derivation above amounts to this: $\varphi_1, \ldots, \varphi_n$ is a derivation iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. $\varphi_i$ is an axiom; or
3. for some $j < i$, $\varphi_j$ is $\psi \rightarrow \varphi_i$, and for some $k < i$, $\varphi_k$ is $\psi$.

The last clause says that $\varphi_i$ follows from $\varphi_j$ ($\psi$) and $\varphi_k$ ($\psi \rightarrow \varphi_i$) by modus ponens. If we can go from 1 to $n$, and each time we find a formula $\varphi_i$ that is either in $\Gamma$, an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivation.

**Definition axd.3 (Derivability).** A formula $\varphi$ is *derivable* from $\Gamma$, written $\Gamma \vdash \varphi$, if there is a derivation from $\Gamma$ ending in $\varphi$.

**Definition axd.4 (Theorems).** A formula $\varphi$ is a *theorem* if there is a derivation of $\varphi$ from the empty set. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\not\vdash \varphi$ if it is not.

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**Bibliography**