axd.1  Rules and Derivations

Axiomatic derivations are perhaps the simplest proof system for logic. A derivation is just a sequence of formulas. To count as a derivation, every formula in the sequence must either be an instance of an axiom, or must follow from one or more formulas that precede it in the sequence by a rule of inference. A derivation derives its last formula.

**Definition axd.1 (Derivability).** If \( \Gamma \) is a set of formulas of \( \mathcal{L} \) then a derivation from \( \Gamma \) is a finite sequence \( \varphi_1, \ldots, \varphi_n \) of formulas where for each \( i \leq n \) one of the following holds:

1. \( \varphi_i \in \Gamma \); or
2. \( \varphi_i \) is an axiom; or
3. \( \varphi_i \) follows from some \( \varphi_j \) (and \( \varphi_k \)) with \( j < i \) (and \( k < i \)) by a rule of inference.

What counts as a correct derivation depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step \( A_i \) in is a correct inference step.

**Definition axd.2 (Rule of inference).** A rule of inference gives a sufficient condition for what counts as a correct inference step in a derivation from \( \Gamma \).

For instance, since any one-element sequence \( \varphi \) with \( \varphi \in \Gamma \) trivially counts as a derivation, the following might be a very simple rule of inference:

If \( \varphi \in \Gamma \), then \( \varphi \) is always a correct inference step in any derivation from \( \Gamma \).

Similarly, if \( \varphi \) is one of the axioms, then \( \varphi \) by itself is a derivation, and so this is also a rule of inference:

If \( \varphi \) is an axiom, then \( \varphi \) is a correct inference step.

It gets more interesting if the rule of inference appeals to formulas that appear before the step considered. The following rule is called *modus ponens*:

If \( \psi \rightarrow \varphi \) and \( \psi \) occur higher up in the derivation, then \( \varphi \) is a correct inference step.

If this is the only rule of inference, then our definition of derivation above amounts to this: \( \varphi_1, \ldots, \varphi_n \) is a derivation iff for each \( i \leq n \) one of the following holds:

1. \( \varphi_i \in \Gamma \); or
2. \( \varphi_i \) is an axiom; or
3. for some \( j < i \), \( \varphi_j \) is \( \psi \rightarrow \varphi_i \), and for some \( k < i \), \( \varphi_k \) is \( \psi \).

The last clause says that \( \varphi_i \) follows from \( \varphi_j (\psi) \) and \( \varphi_k (\psi \rightarrow \varphi_i) \) by modus ponens. If we can go from 1 to \( n \), and each time we find a formula \( \varphi_i \) that is either in \( \Gamma \), an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivation.

**Definition axd.3** (Derivability). A formula \( \varphi \) is *derivable* from \( \Gamma \), written \( \Gamma \vdash \varphi \), if there is a derivation from \( \Gamma \) ending in \( \varphi \).

**Definition axd.4** (Theorems). A formula \( \varphi \) is a *theorem* if there is a derivation of \( \varphi \) from the empty set. We write \( \vdash \varphi \) if \( \varphi \) is a theorem and \( \nvdash \varphi \) if it is not.

Photo Credits

Bibliography