

## axd.1 Rules and Derivations

fol:axd:rul: Axiomatic **derivations** are perhaps the simplest **derivation** system for logic. explanation  
sec A **derivation** is just a sequence of **formulas**. To count as a **derivation**, every **formula** in the sequence must either be an instance of an axiom, or must follow from one or more **formulas** that precede it in the sequence by a rule of inference. A **derivation derives** its last **formula**.

**Definition axd.1 (Derivability).** If  $\Gamma$  is a set of **formulas** of  $\mathcal{L}$  then a *derivation* from  $\Gamma$  is a finite sequence  $\varphi_1, \dots, \varphi_n$  of **formulas** where for each  $i \leq n$  one of the following holds:

1.  $\varphi_i \in \Gamma$ ; or
2.  $\varphi_i$  is an axiom; or
3.  $\varphi_i$  follows from some  $\varphi_j$  (and  $\varphi_k$ ) with  $j < i$  (and  $k < i$ ) by a rule of inference.

What counts as a correct **derivation** depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step  $A_i$  in a **derivation** is a correct inference step.

**Definition axd.2 (Rule of inference).** A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a **derivation** from  $\Gamma$ .

For instance, since any one-element sequence  $\varphi$  with  $\varphi \in \Gamma$  trivially counts as a **derivation**, the following might be a very simple rule of inference:

If  $\varphi \in \Gamma$ , then  $\varphi$  is always a correct inference step in any **derivation** from  $\Gamma$ .

Similarly, if  $\varphi$  is one of the axioms, then  $\varphi$  by itself is a **derivation**, and so this is also a rule of inference:

If  $\varphi$  is an axiom, then  $\varphi$  is a correct inference step.

It gets more interesting if the rule of inference appeals to **formulas** that appear before the step considered. The following rule is called *modus ponens*:

If  $\psi \rightarrow \varphi$  and  $\psi$  occur higher up in the **derivation**, then  $\varphi$  is a correct inference step.

If this is the only rule of inference, then our definition of **derivation** above amounts to this:  $\varphi_1, \dots, \varphi_n$  is a **derivation** iff for each  $i \leq n$  one of the following holds:

1.  $\varphi_i \in \Gamma$ ; or
2.  $\varphi_i$  is an axiom; or

3. for some  $j < i$ ,  $\varphi_j$  is  $\psi \rightarrow \varphi_i$ , and for some  $k < i$ ,  $\varphi_k$  is  $\psi$ .

The last clause says that  $\varphi_i$  follows from  $\varphi_j$  ( $\psi$ ) and  $\varphi_k$  ( $\psi \rightarrow \varphi_i$ ) by modus ponens. If we can go from 1 to  $n$ , and each time we find a formula  $\varphi_i$  that is either in  $\Gamma$ , an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct **derivation**.

**Definition axd.3 (Derivability).** A formula  $\varphi$  is *derivable* from  $\Gamma$ , written  $\Gamma \vdash \varphi$ , if there is a **derivation** from  $\Gamma$  ending in  $\varphi$ .

**Definition axd.4 (Theorems).** A formula  $\varphi$  is a *theorem* if there is a **derivation** of  $\varphi$  from the empty set. We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

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## Bibliography