

axd.1 Rules and Derivations

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sec Axiomatic **derivations** are perhaps the simplest proof system for logic. A explanation **derivation** is just a sequence of **formulas**. To count as a **derivation**, every **formula** in the sequence must either be an instance of an axiom, or must follow from one or more **formulas** that precede it in the sequence by a rule of inference. A **derivation derives** its last **formula**.

Definition axd.1 (Derivability). If Γ is a set of **formulas** of \mathcal{L} then a **derivation** from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of **formulas** where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct **derivation** depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in is a correct inference step.

Definition axd.2 (Rule of inference). A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a **derivation** from Γ .

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a **derivation**, the following might be a very simple rule of inference:

If $\varphi \in \Gamma$, then φ is always a correct inference step in any **derivation** from Γ .

Similarly, if φ is one of the axioms, then φ by itself is a **derivation**, and so this is also a rule of inference:

If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to **formulas** that appear before the step considered. The following rule is called *modus ponens*:

If $\psi \rightarrow \varphi$ and ψ occur higher up in the **derivation**, then φ is a correct inference step.

If this is the only rule of inference, then our definition of **derivation** above amounts to this: $\varphi_1, \dots, \varphi_n$ is a **derivation** iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or

3. for some $j < i$, φ_j is $\psi \rightarrow \varphi_i$, and for some $k < i$, φ_k is ψ .

The last clause says that φ_i follows from φ_j (ψ) and φ_k ($\psi \rightarrow \varphi_i$) by modus ponens. If we can go from 1 to n , and each time we find a formula φ_i that is either in Γ , an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct **derivation**.

Definition axd.3 (Derivability). A formula φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a **derivation** from Γ ending in φ .

Definition axd.4 (Theorems). A formula φ is a *theorem* if there is a **derivation** of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

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Bibliography