axd.1 Rules and Derivations

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Axiomatic derivations are perhaps the simplest proof system for logic. A derivation is just a sequence of formulas. To count as a derivation, every formula in the sequence must either be an instance of an axiom, or must follow from one or more formulas that precede it in the sequence by a rule of inference. A derivation derives its last formula. explanation

Definition axd.1 (Derivability)**.** If Γ is a set of formulas of $\mathcal{L}$ then a *derivation* from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct derivation depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in is a correct inference step.

Definition axd.2 (Rule of inference)**.** A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a derivation from Γ.

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a derivation, the following might be a very simple rule of inference:

> If $\varphi \in \Gamma$, then φ is always a correct inference step in any derivation from Γ.

Similarly, if φ is one of the axioms, then φ by itself is a derivation, and so this is also a rule of inference:

> If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to formulas that appear before the step considered. The following rule is called *modus ponens:*

> If $\psi \rightarrow \varphi$ and ψ occur higher up in the derivation, then φ is a correct inference step.

If this is the only rule of inference, then our definition of derivation above amounts to this: $\varphi_1, \dots, \varphi_n$ is a derivation iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or

3. for some $j < i$, φ_j is $\psi \rightarrow \varphi_i$, and for some $k < i$, φ_k is ψ.

The last clause says that φ_i follows from φ_j (ψ) and φ_k ($\psi \rightarrow \varphi_i$) by modus ponens. If we can go from 1 to n, and each time we find a formula φ_i that is either in Γ, an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivation.

Definition axd.3 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition axd.4 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

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Bibliography