axd.1 Examples of Derivations

fol:axd:pro:
sec

Example axd.1. Suppose we want to prove $(\lnot\theta \lor \alpha) \to (\theta \to \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to derive it. Our only rule is MP, which given φ and $\varphi \to \psi$ allows us to justify ψ. One strategy would be to use **??** with φ being $\lnot\theta$, ψ being α, and χ being $\theta \to \alpha$, i.e., the instance

$$(\lnot\theta \to (\theta \to \alpha)) \to ((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\lnot\theta \to (\theta \to \alpha)$ is an instance of **??**, and $\alpha \to (\theta \to \alpha)$ is an instance of **??**. So our derivation is:

1.	$\lnot\theta \to (\theta \to \alpha)$	**??**
2.	$(\lnot\theta \to (\theta \to \alpha)) \to$ $((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha)))$	**??**
3.	$((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha))$	1, 2, MP
4.	$\alpha \to (\theta \to \alpha)$	**??**
5.	$(\lnot\theta \lor \alpha) \to (\theta \to \alpha)$	3, 4, MP

fol:axd:pro:
ex:identity

Example axd.2. Let's try to find a derivation of $\theta \to \theta$. It is not an instance of an axiom, so we have to use MP to derive it. **??** is an axiom of the form $\varphi \to \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \to \theta$, since this is what we want to derive. That means φ would also have to be θ, i.e., we might look at this instance of **??**:

$$\theta \to (\theta \to \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ. But in our case, that would be θ, and we won't be able to derive θ by itself. So we need a different strategy.

The other axiom involving just $\to$ is **??**, i.e.,

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of **??** where $\varphi \to \chi$ is $\theta \to \theta$, the formula we are aiming for. Then of course, φ and χ are both θ. How should we pick ψ so that both $\varphi \to (\psi \to \chi)$ and $\varphi \to \psi$, i.e., in our case $\theta \to (\psi \to \theta)$ and $\theta \to \psi$, are also derivable? Well, the first of these is already an instance of **??**, whatever we decide ψ to be. And $\theta \to \psi$ would be another instance of **??** if ψ were $(\theta \to \theta)$. So, our derivation is:

1. $\theta \to ((\theta \to \theta) \to \theta)$ **??**
2. $(\theta \to ((\theta \to \theta) \to \theta)) \to$
   $((\theta \to (\theta \to \theta)) \to (\theta \to \theta))$ **??**
3. $(\theta \to (\theta \to \theta)) \to (\theta \to \theta)$ 1, 2, MP
4. $\theta \to (\theta \to \theta)$ **??**
5. $\theta \to \theta$ 3, 4, MP

Example axd.3. Sometimes we want to show that there is a derivation of some formula from some other formulas Γ. For instance, let's show that we can derive $\varphi \to \chi$ from $\Gamma = \{\varphi \to \psi, \psi \to \chi\}$. fol:axd:pro:ex:chain

1. $\varphi \to \psi$ Hyp
2. $\psi \to \chi$ Hyp
3. $(\psi \to \chi) \to (\varphi \to (\psi \to \chi))$ **??**
4. $\varphi \to (\psi \to \chi)$ 2, 3, MP
5. $(\varphi \to (\psi \to \chi)) \to$
   $((\varphi \to \psi) \to (\varphi \to \chi))$ **??**
6. $((\varphi \to \psi) \to (\varphi \to \chi))$ 4, 5, MP
7. $\varphi \to \chi$ 1, 6, MP

The lines labelled "Hyp" (for "hypothesis") indicate that the formula on that line is an element of Γ.

Proposition axd.4. *If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$, then $\Gamma \vdash \varphi \to \chi$* *fol:axd:pro:prop:chain*

Proof. Suppose $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$. Then there is a derivation of $\varphi \to \psi$ from Γ; and a derivation of $\psi \to \chi$ from Γ as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of $\varphi \to \chi$—which is the last line of the new derivation—from Γ. Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of $\varphi \to \psi$, and reference to line number 1 by reference to the last line of the derivation of $B \to \chi$. □

Problem axd.1. Show that the following hold by exhibiting derivations from the axioms:

1. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
3. $\neg(\varphi \vee \psi) \to \neg\varphi$

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Bibliography