

axd.1 Examples of Derivations

fol:axd:pro:
sec

Example axd.1. Suppose we want to prove $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to **derive** it. Our only rule is MP, which given φ and $\varphi \rightarrow \psi$ allows us to justify ψ . One strategy would be to use ?? with φ being $\neg\theta$, ψ being α , and χ being $\theta \rightarrow \alpha$, i.e., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ is an instance of ??, and $\alpha \rightarrow (\theta \rightarrow \alpha)$ is an instance of ?. So our derivation is:

- | | | |
|----|---|----------|
| 1. | $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ | ?? |
| 2. | $(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$ $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ | ?? |
| 3. | $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ | 1, 2, MP |
| 4. | $\alpha \rightarrow (\theta \rightarrow \alpha)$ | ?? |
| 5. | $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$ | 3, 4, MP |

fol:axd:pro:
ex:identity

Example axd.2. Let's try to find a **derivation** of $\theta \rightarrow \theta$. It is not an instance of an axiom, so we have to use MP to **derive** it. ?? is an axiom of the form $\varphi \rightarrow \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \rightarrow \theta$, since this is what we want to **derive**. That means φ would also have to be θ , i.e., we might look at this instance of ??:

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ . But in our case, that would be θ , and we won't be able to **derive** θ by itself. So we need a different strategy.

The other axiom involving just \rightarrow is ??, i.e.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of ?? where $\varphi \rightarrow \chi$ is $\theta \rightarrow \theta$, the **formula** we are aiming for. Then of course, φ and χ are both θ . How should we pick ψ so that both $\varphi \rightarrow (\psi \rightarrow \chi)$ and $\varphi \rightarrow \psi$, i.e., in our case $\theta \rightarrow (\psi \rightarrow \theta)$ and $\theta \rightarrow \psi$, are also **derivable**? Well, the first of these is already an instance of ??, whatever we decide ψ to be. And $\theta \rightarrow \psi$ would be another instance of ?? if ψ were $(\theta \rightarrow \theta)$. So, our derivation is:

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|----|---|----------|
| 1. | $\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$ | ?? |
| 2. | $(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$ $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$ | ?? |
| 3. | $(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$ | 1, 2, MP |
| 4. | $\theta \rightarrow (\theta \rightarrow \theta)$ | ?? |
| 5. | $\theta \rightarrow \theta$ | 3, 4, MP |

Example axd.3. Sometimes we want to show that there is a derivation of some formula from some other formulas Γ . For instance, let's show that we can derive $\varphi \rightarrow \chi$ from $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$. fol:axd:pro:
ex:chain

- | | | |
|----|--|----------|
| 1. | $\varphi \rightarrow \psi$ | HYP |
| 2. | $\psi \rightarrow \chi$ | HYP |
| 3. | $(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ | ?? |
| 4. | $\varphi \rightarrow (\psi \rightarrow \chi)$ | 2, 3, MP |
| 5. | $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$ $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ | ?? |
| 6. | $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ | 4, 5, MP |
| 7. | $\varphi \rightarrow \chi$ | 1, 6, MP |

The lines labelled “HYP” (for “hypothesis”) indicate that the formula on that line is an element of Γ .

Proposition axd.4. *If $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$, then $\Gamma \vdash \varphi \rightarrow \chi$* fol:axd:pro:
prop:chain

Proof. Suppose $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$. Then there is a derivation of $\varphi \rightarrow \psi$ from Γ ; and a derivation of $\psi \rightarrow \chi$ from Γ as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of $\varphi \rightarrow \chi$ —which is the last line of the new derivation—from Γ . Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of $\varphi \rightarrow \psi$, and reference to line number 1 by reference to the last line of the derivation of $\psi \rightarrow \chi$. □

Problem axd.1. Show that the following hold by exhibiting derivations from the axioms:

1. $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3. $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

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Bibliography