axd.1 Derivability and the Propositional Connectives

fol:axd:ppr: sec

We establish that the derivability relation $\vdash$ of axiomatic deduction is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \rightarrow \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. explanation

fol:axd:ppr: prop:provability-land

Proposition axd.1.

fol:axd:ppr: prop:provability-land-left

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$

fol:axd:ppr: prop:provability-land-right

2. $\varphi, \psi \vdash \varphi \land \psi$.

Proof. 1. From **??** and **??** by modus ponens.

2. From **??** by two applications of modus ponens. □

fol:axd:ppr: prop:provability-lor

Proposition axd.2.

1. $\varphi \lor \psi, \lnot\varphi, \lnot\psi$ *is inconsistent.*

2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. From **??** we get $\vdash \lnot\varphi \rightarrow (\varphi \rightarrow \bot)$ and $\vdash \lnot\psi \rightarrow (\psi \rightarrow \bot)$. So by the deduction theorem, we have $\{\lnot\varphi\} \vdash \varphi \rightarrow \bot$ and $\{\lnot\psi\} \vdash \psi \rightarrow \bot$. From **??** we get $\{\lnot\varphi, \lnot\psi\} \vdash (\varphi \lor \psi) \rightarrow \bot$. By the deduction theorem, $\{\varphi \lor \psi, \lnot\varphi, \lnot\psi\} \vdash \bot$.

2. From **??** and **??** by modus ponsens. □

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Proposition axd.3.

fol:axd:ppr: prop:provability-lif-left

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.

fol:axd:ppr: prop:provability-lif-right

2. *Both* $\lnot\varphi \vdash \varphi \rightarrow \psi$ *and* $\psi \vdash \varphi \rightarrow \psi$.

Proof. 1. We can derive:

1.	φ	Hyp
2.	$\varphi \rightarrow \psi$	Hyp
3.	ψ	1, 2, mp

2. By **??** and **??** and the deduction theorem, respectively. □

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Bibliography