We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition axd.1.** If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

*Proof.* If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By $\text{??}$, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By $\text{??}$, $\Gamma \vdash \bot$, i.e., $\Gamma$ is inconsistent. $\Box$

**Proposition axd.2.** $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

*Proof.* First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg \varphi\} \vdash \varphi$ by $\text{??}$. $\Gamma \cup \{\neg \varphi\} \vdash \neg \varphi$ by $\text{??}$. We also have $\vdash \neg \varphi \rightarrow (\varphi \rightarrow \bot)$ by $\text{??}$. So by two applications of $\text{??}$, we have $\Gamma \cup \{\neg \varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg \varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg \varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg \varphi \rightarrow \bot$. $\Gamma \vdash (\neg \varphi \rightarrow \bot) \rightarrow \neg \neg \varphi$ by $\text{??}$, so $\Gamma \vdash \neg \neg \varphi$ by $\text{??}$. Since $\Gamma \vdash \neg \neg \varphi \rightarrow \varphi$ ($\text{??}$), we have $\Gamma \vdash \varphi$ by $\text{??}$ again. $\Box$

**Problem axd.1.** Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

**Proposition axd.3.** If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

*Proof.* $\Gamma \vdash \neg \varphi \rightarrow (\varphi \rightarrow \bot)$ by $\text{??}$. $\Gamma \vdash \bot$ by two applications of $\text{??}$. $\Box$

**Proposition axd.4.** If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

*Proof.* Exercise. $\Box$

**Problem axd.2.** Prove Proposition axd.4

---

**Photo Credits**

**Bibliography**