We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition axd.1.** If $\Gamma \vdash \varphi$ and $\Gamma \cup \{ \varphi \}$ is inconsistent, then $\Gamma$ is inconsistent.

*Proof.* If $\Gamma \cup \{ \varphi \}$ is inconsistent, then $\Gamma \cup \{ \varphi \} \vdash \bot$. By $\varphi$, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{ \varphi \}$. By $\psi$, $\Gamma \vdash \bot$, i.e., $\Gamma$ is inconsistent.

**Proposition axd.2.** $\Gamma \vdash \varphi$ iff $\Gamma \cup \{ \lnot \varphi \}$ is inconsistent.

*Proof.* First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{ \lnot \varphi \} \vdash \varphi$ by $\varphi$. $\Gamma \cup \{ \lnot \varphi \} \vdash \lnot \varphi$ by $\lnot \varphi$. We also have $\vdash \lnot \varphi \rightarrow (\varphi \rightarrow \bot)$ by $\varphi$. So by two applications of $\varphi$, we have $\Gamma \vdash \bot$.

Now assume $\Gamma \cup \{ \lnot \varphi \}$ is inconsistent, i.e., $\Gamma \cup \{ \lnot \varphi \} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \lnot \varphi \rightarrow \bot$. $\Gamma \vdash (\lnot \varphi \rightarrow \bot) \rightarrow \lnot \varphi$ by $\bot$, so $\Gamma \vdash \lnot \varphi$ by $\bot$. Since $\Gamma \vdash \lnot \varphi \rightarrow \varphi$ (?), we have $\Gamma \vdash \varphi$ by $\bot$ again.

**Problem axd.1.** Prove that $\Gamma \vdash \lnot \varphi$ iff $\Gamma \cup \{ \varphi \}$ is inconsistent.

**Proposition axd.3.** If $\Gamma \vdash \varphi$ and $\lnot \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

*Proof.* $\Gamma \vdash \lnot \varphi \rightarrow (\varphi \rightarrow \bot)$ by $\bot$. $\Gamma \vdash \bot$ by two applications of $\bot$.

**Proposition axd.4.** If $\Gamma \cup \{ \varphi \}$ and $\Gamma \cup \{ \lnot \varphi \}$ are both inconsistent, then $\Gamma$ is inconsistent.

*Proof.* Exercise.

**Problem axd.2.** Prove Proposition axd.4

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**Bibliography**