axd.1 Derivability and Consistency

fol:axd:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:axd:prv: prop:provability-contr

Proposition axd.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By **??**, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By **??**, $\Gamma \vdash \bot$, i.e., Γ is inconsistent. □

fol:axd:prv: prop:prov-incons

Proposition axd.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by **??**. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by **??**. We also have $\vdash \neg\varphi \to (\varphi \to \bot)$ by **??**. So by two applications of **??**, we have $\Gamma \cup \{\neg\varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg\varphi \to \bot$. $\Gamma \vdash (\neg\varphi \to \bot) \to \neg\neg\varphi$ by **??**, so $\Gamma \vdash \neg\neg\varphi$ by **??**. Since $\Gamma \vdash \neg\neg\varphi \to \varphi$ (**??**), we have $\Gamma \vdash \varphi$ by **??** again. □

Problem axd.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:axd:prv: prop:explicit-inc

Proposition axd.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. $\Gamma \vdash \neg\varphi \to (\varphi \to \bot)$ by **??**. $\Gamma \vdash \bot$ by two applications of **??**. □

fol:axd:prv: prop:provability-exhaustive

Proposition axd.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. Exercise. □

Problem axd.2. Prove Proposition axd.4

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Bibliography