

axd.1 Derivability and Consistency

fol:axd:prv:
sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:axd:prv:
prop:provability-contr **Proposition axd.1.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \perp$. By ??, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By ??, $\Gamma \vdash \perp$, i.e., Γ is inconsistent. \square

fol:axd:prv:
prop:prov-incons **Proposition axd.2.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by ?. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by ?. We also have $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by ?. So by two applications of ?, we have $\Gamma \cup \{\neg\varphi\} \vdash \perp$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \perp$. By the deduction theorem, $\Gamma \vdash \neg\varphi \rightarrow \perp$. $\Gamma \vdash (\neg\varphi \rightarrow \perp) \rightarrow \neg\neg\varphi$ by ?, so $\Gamma \vdash \neg\neg\varphi$ by ?. Since $\Gamma \vdash \neg\neg\varphi \rightarrow \varphi$ (?), we have $\Gamma \vdash \varphi$ by ? again. \square

Problem axd.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:axd:prv:
prop:explicit-inc **Proposition axd.3.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. $\Gamma \vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by ?. $\Gamma \vdash \perp$ by two applications of ?. \square

fol:axd:prv:
prop:provability-exhaustive **Proposition axd.4.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. Exercise. \square

Problem axd.2. Prove [Proposition axd.4](#)

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Bibliography