

axd.1 Proof-Theoretic Notions

fol:axd:ptn:
sec Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. explanation These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

Definition axd.1 (Derivability). A formula φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a **derivation** from Γ ending in φ .

Definition axd.2 (Theorems). A formula φ is a *theorem* if there is a **derivation** of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition axd.3 (Consistency). A set Γ of **formulas** is *consistent* if and only if $\Gamma \not\vdash \perp$; it is *inconsistent* otherwise.

fol:axd:ptn:
prop:reflexivity **Proposition axd.4 (Reflexivity).** If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Proof. The **formula** φ by itself is a **derivation** of φ from Γ . □

fol:axd:ptn:
prop:monotonicity **Proposition axd.5 (Monotonicity).** If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.

Proof. Any **derivation** of φ from Γ is also a **derivation** of φ from Δ . □

fol:axd:ptn:
prop:transitivity **Proposition axd.6 (Transitivity).** If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.

Proof. Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a **derivation** $\psi_1, \dots, \psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivation will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a **derivation** of φ from Γ , i.e., a **derivation** $\varphi_1, \dots, \varphi_k = \varphi$ where every φ_i is an axiom, an **element** of Γ , or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct **derivation** of ψ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

fol:axd:ptn:
prop:incons **Proposition axd.7.** Γ is *inconsistent* iff $\Gamma \vdash \varphi$ for every φ .

Proof. Exercise. □

Problem axd.1. Prove **Proposition axd.7**.

Proposition axd.8 (Compactness).

*fol:axd:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, an element of Γ or follows from previous formulas by modus ponens. Take Γ_0 to be those φ_i which are in Γ . Then the derivation is likewise a derivation from Γ_0 , and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. □

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Bibliography