axd.1 The Deduction Theorem

fol:axd:ded: sec

As we've seen, giving derivations in an axiomatic system is cumbersome, and derivations may be hard to find. Rather than actually write out long lists of formulas, it is generally easier to argue that such derivations exist, by making use of a few simple results. We've already established three such results: **??** says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. **??** says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And **??** implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens:

fol:axd:ded: prop:mp

Proposition axd.1. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.*

Proof. We have that $\{\varphi, \varphi \to \psi\} \vdash \psi$:

1.	φ	Hyp.
2.	$\varphi \to \psi$	Hyp.
3.	ψ	1, 2, MP

By **??**, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

fol:axd:ded: thm:deduction-thm

Theorem axd.2 (Deduction Theorem)**.** *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \to \psi$.*

Proof. The "if" direction is immediate. If $\Gamma \vdash \varphi \to \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$ by **??**. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by **??**. So, by Proposition axd.1, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the derivation of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every derivation of length 1. A derivation of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \to (\varphi \to \psi)$ by **??**, and Proposition axd.1 gives $\Gamma \vdash \varphi \to \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \to \psi$ because then last sentence $\varphi \to \psi$ is the same as $\varphi \to \varphi$, and we have derived that in **??**.

For the inductive step, suppose a derivation of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the derivation are $\chi \to \psi$ and χ, for some formula χ, i.e., $\Gamma \cup \{\varphi\} \vdash \chi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\Gamma \vdash \varphi \to (\chi \to \psi);$$
$$\Gamma \vdash \varphi \to \chi.$$

But also

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi)),$$

by **??**, and two applications of Proposition axd.1 give $\Gamma \vdash \varphi \to \psi$, as required. □

Notice how **??** and **??** were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about derivability, which we leave as exercises.

Proposition axd.3. *fol:axd:ded:prop:derivfacts*

1. $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)$; *fol:axd:ded:derivfacts:a*
2. *If* $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ *then* $\Gamma \cup \{\psi\} \vdash \varphi$ *(Contraposition);* *fol:axd:ded:derivfacts:b*
3. $\{\varphi, \neg\varphi\} \vdash \psi$ *(Ex Falso Quodlibet, Explosion);* *fol:axd:ded:derivfacts:c*
4. $\{\neg\neg\varphi\} \vdash \varphi$ *(Double Negation Elimination);* *fol:axd:ded:derivfacts:d*
5. *If* $\Gamma \vdash \neg\neg\varphi$ *then* $\Gamma \vdash \varphi$*;* *fol:axd:ded:derivfacts:e*

Problem axd.1. Prove Proposition axd.3

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Bibliography