

## axd.1 The Deduction Theorem

As we've seen, giving **derivations** in an axiomatic system is cumbersome, and **derivations** may be hard to find. Rather than actually write out long lists of **formulas**, it is generally easier to argue that such **derivations** exist, by making use of a few simple results. We've already established three such results: ?? says we can always assert that  $\Gamma \vdash \varphi$  when we know that  $\varphi \in \Gamma$ . ?? says that if  $\Gamma \vdash \varphi$  then also  $\Gamma \cup \{\psi\} \vdash \varphi$ . And ?? implies that if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . Here's another simple result, a "meta"-version of modus ponens:

**Proposition axd.1.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \varphi \rightarrow \psi$ , then  $\Gamma \vdash \psi$ .*

*Proof.* We have that  $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$ :

1.  $\varphi$  Hyp.
2.  $\varphi \rightarrow \psi$  Hyp.
3.  $\psi$  1, 2, MP

By ??,  $\Gamma \vdash \psi$ . □

The most important result we'll use in this context is the deduction theorem:

**Theorem axd.2 (Deduction Theorem).**  *$\Gamma \cup \{\varphi\} \vdash \psi$  if and only if  $\Gamma \vdash \varphi \rightarrow \psi$ .*

*Proof.* The "if" direction is immediate. If  $\Gamma \vdash \varphi \rightarrow \psi$  then also  $\Gamma \cup \{\varphi\} \vdash \varphi \rightarrow \psi$  by ??. Also,  $\Gamma \cup \{\varphi\} \vdash \varphi$  by ??. So, by **Proposition axd.1**,  $\Gamma \cup \{\varphi\} \vdash \psi$ .

For the "only if" direction, we proceed by induction on the length of the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$ .

For the induction basis, we prove the claim for every **derivation** of length 1. A **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  of length 1 consists of  $\psi$  by itself; and if it is correct  $\psi$  is either  $\in \Gamma \cup \{\varphi\}$  or is an axiom. If  $\psi \in \Gamma$  or is an axiom, then  $\Gamma \vdash \psi$ . We also have that  $\Gamma \vdash \psi \rightarrow (\varphi \rightarrow \psi)$  by ??, and **Proposition axd.1** gives  $\Gamma \vdash \varphi \rightarrow \psi$ . If  $\psi \in \{\varphi\}$  then  $\Gamma \vdash \varphi \rightarrow \psi$  because then last **sentence**  $\varphi \rightarrow \psi$  is the same as  $\varphi \rightarrow \varphi$ , and we have **derived** that in ??.

For the inductive step, suppose a **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  ends with a step  $\psi$  which is justified by modus ponens. (If it is not justified by modus ponens,  $\psi \in \Gamma$ ,  $\psi \equiv \varphi$ , or  $\psi$  is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the **derivation** are  $\chi \rightarrow \psi$  and  $\chi$ , for some **formula**  $\chi$ , i.e.,  $\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \psi$  and  $\Gamma \cup \{\varphi\} \vdash \chi$ , and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\begin{aligned} \Gamma \vdash \varphi \rightarrow (\chi \rightarrow \psi); \\ \Gamma \vdash \varphi \rightarrow \chi. \end{aligned}$$

But also

$$\Gamma \vdash (\varphi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)),$$

by ??, and two applications of **Proposition axd.1** give  $\Gamma \vdash \varphi \rightarrow \psi$ , as required. □

Notice how ?? and ?? were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about **derivability**, which we leave as exercises.

**Proposition axd.3.**

1.  $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$ ;
2. *If  $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$  then  $\Gamma \cup \{\psi\} \vdash \varphi$  (Contraposition);*
3.  $\{\varphi, \neg\varphi\} \vdash \psi$  (*Ex Falso Quodlibet, Explosion*);
4.  $\{\neg\neg\varphi\} \vdash \varphi$  (*Double Negation Elimination*);
5. *If  $\Gamma \vdash \neg\neg\varphi$  then  $\Gamma \vdash \varphi$ ;*

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prop:derifacts  
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**Problem axd.1.** Prove [Proposition axd.3](#)

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## Bibliography