axd.1 The Deduction Theorem with Quantifiers

**Theorem axd.1 (Deduction Theorem).** If $\Gamma \cup \{\varphi\} \vdash \psi$ and then $\Gamma \vdash \varphi \rightarrow \psi$.

**Proof.** We again proceed by induction on the length of the derivation of $\psi$ from $\Gamma \cup \{\varphi\}$.

The proof of the induction basis is identical to that in the proof of ??.

For the inductive step, suppose again that the derivation of $\psi$ from $\Gamma \cup \{\varphi\}$ ends with a step $\psi$ which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of ?? . If the inference rule is QR, we know that $\psi \equiv \chi \rightarrow \forall x \theta(x)$ and a formula of the form $\chi \rightarrow \theta(a)$ appears earlier in the derivation, where $a$ does not occur in $\chi$, $\varphi$, or $\Gamma$. We thus have that

$$
\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \theta(a)
$$

and the induction hypothesis applies, i.e., we have that

$$
\Gamma \vdash \varphi \rightarrow \theta(a)
$$

By

$$
\vdash (\varphi \rightarrow (\chi \rightarrow \theta(a))) \rightarrow ((\varphi \land \chi) \rightarrow \theta(a))
$$

and modus ponens we get

$$
\Gamma \vdash (\varphi \land \chi) \rightarrow \theta(a).
$$

Since the eigenvariale condition still applies, we can add a step to this derivation justified by QR, and get:

$$
\Gamma \vdash (\varphi \land \chi) \rightarrow \forall x \theta(x)
$$

We also have

$$
\vdash ((\varphi \land \chi) \rightarrow \forall x \theta(x)) \rightarrow (\varphi \rightarrow (\chi \rightarrow \forall x \theta(x)))
$$

so by modus ponens,

$$
\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \forall x \theta(x))
$$

i.e., $\Gamma \vdash \psi$.

We leave the case where $\psi$ is justified by the rule QR, but is of the form $\exists x \theta(x) \rightarrow \chi$, as an exercise. \qed
Problem axd.1. Complete the proof of Theorem axd.1.

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Bibliography