

## Chapter udf

# Axiomatic Derivations

No effort has been made yet to ensure that the material in this chapter respects various tags indicating which connectives and quantifiers are primitive or defined: all are assumed to be primitive. If the FOL tag is true, we produce a version with quantifiers, otherwise without.

### axd.1 Rules and Derivations

fol:axd:rul:  
sec Axiomatic **derivations** are perhaps the simplest proof system for logic. A explanation **derivation** is just a sequence of **formulas**. To count as a **derivation**, every **formula** in the sequence must either be an instance of an axiom, or must follow from one or more **formulas** that precede it in the sequence by a rule of inference. A **derivation derives** its last **formula**.

**Definition axd.1 (Derivability).** If  $\Gamma$  is a set of **formulas** of  $\mathcal{L}$  then a **derivation** from  $\Gamma$  is a finite sequence  $\varphi_1, \dots, \varphi_n$  of **formulas** where for each  $i \leq n$  one of the following holds:

1.  $\varphi_i \in \Gamma$ ; or
2.  $\varphi_i$  is an axiom; or
3.  $\varphi_i$  follows from some  $\varphi_j$  (and  $\varphi_k$ ) with  $j < i$  (and  $k < i$ ) by a rule of inference.

What counts as a correct **derivation** depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step  $A_i$  in is a correct inference step.

**Definition axd.2 (Rule of inference).** A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a **derivation** from  $\Gamma$ .

For instance, since any one-element sequence  $\varphi$  with  $\varphi \in \Gamma$  trivially counts as a **derivation**, the following might be a very simple rule of inference:

If  $\varphi \in \Gamma$ , then  $\varphi$  is always a correct inference step in any **derivation** from  $\Gamma$ .

Similarly, if  $\varphi$  is one of the axioms, then  $\varphi$  by itself is a **derivation**, and so this is also a rule of inference:

If  $\varphi$  is an axiom, then  $\varphi$  is a correct inference step.

It gets more interesting if the rule of inference appeals to **formulas** that appear before the step considered. The following rule is called *modus ponens*:

If  $\psi \rightarrow \varphi$  and  $\psi$  occur higher up in the **derivation**, then  $\varphi$  is a correct inference step.

If this is the only rule of inference, then our definition of **derivation** above amounts to this:  $\varphi_1, \dots, \varphi_n$  is a **derivation** iff for each  $i \leq n$  one of the following holds:

1.  $\varphi_i \in \Gamma$ ; or
2.  $\varphi_i$  is an axiom; or
3. for some  $j < i$ ,  $\varphi_j$  is  $\psi \rightarrow \varphi_i$ , and for some  $k < i$ ,  $\varphi_k$  is  $\psi$ .

The last clause says that  $\varphi_i$  follows from  $\varphi_j$  ( $\psi$ ) and  $\varphi_k$  ( $\psi \rightarrow \varphi_i$ ) by modus ponens. If we can go from 1 to  $n$ , and each time we find a **formula**  $\varphi_i$  that is either in  $\Gamma$ , an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct **derivation**.

**Definition axd.3 (Derivability).** A **formula**  $\varphi$  is *derivable* from  $\Gamma$ , written  $\Gamma \vdash \varphi$ , if there is a **derivation** from  $\Gamma$  ending in  $\varphi$ .

**Definition axd.4 (Theorems).** A **formula**  $\varphi$  is a *theorem* if there is a **derivation** of  $\varphi$  from the empty set. We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

## axd.2 Axiom and Rules for the Propositional Connectives

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**Definition axd.5** (Axioms). The set of  $Ax_0$  of *axioms* for the propositional connectives comprises all **formulas** of the following forms:

<code>fol:axd:prp:</code>	$(\varphi \wedge \psi) \rightarrow \varphi$	(axd.1)
<code>ax:land1</code> <code>fol:axd:prp:</code>	$(\varphi \wedge \psi) \rightarrow \psi$	(axd.2)
<code>ax:land2</code> <code>fol:axd:prp:</code>	$\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$	(axd.3)
<code>ax:land3</code> <code>fol:axd:prp:</code>	$\varphi \rightarrow (\varphi \vee \psi)$	(axd.4)
<code>ax:lor1</code> <code>fol:axd:prp:</code>	$\varphi \rightarrow (\psi \vee \varphi)$	(axd.5)
<code>ax:lor2</code> <code>fol:axd:prp:</code>	$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$	(axd.6)
<code>ax:lor3</code> <code>fol:axd:prp:</code>	$\varphi \rightarrow (\psi \rightarrow \varphi)$	(axd.7)
<code>ax:lif1</code> <code>fol:axd:prp:</code>	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	(axd.8)
<code>ax:lif2</code> <code>fol:axd:prp:</code>	$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$	(axd.9)
<code>ax:lnot1</code> <code>fol:axd:prp:</code>	$\neg\varphi \rightarrow (\varphi \rightarrow \psi)$	(axd.10)
<code>ax:lnot2</code> <code>fol:axd:prp:</code>	$\top$	(axd.11)
<code>ax:ltrue</code> <code>fol:axd:prp:</code>	$\perp \rightarrow \varphi$	(axd.12)
<code>ax:lfalse1</code> <code>fol:axd:prp:</code>	$(\varphi \rightarrow \perp) \rightarrow \neg\varphi$	(axd.13)
<code>ax:lfalse2</code> <code>fol:axd:prp:</code>	$\neg\neg\varphi \rightarrow \varphi$	(axd.14)
<code>ax:dne</code>		

**Definition axd.6** (Modus ponens). If  $\psi$  and  $\psi \rightarrow \varphi$  already occur in a derivation, then  $\varphi$  is a correct inference step.

We'll abbreviate the rule modus ponens as “MP.”

### axd.3 Axioms and Rules for Quantifiers

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`sec`

**Definition axd.7** (Axioms for quantifiers). The *axioms* governing quantifiers are all instances of the following:

$$\forall x \psi \rightarrow \psi(t), \quad (\text{axd.15})$$

$$\psi(t) \rightarrow \exists x \psi. \quad (\text{axd.16})$$

`ax:q2`

for any ground term  $t$ .

**Definition axd.8** (Rules for quantifiers).

If  $\psi \rightarrow \varphi(a)$  already occurs in the **derivation** and  $a$  does not occur in  $\Gamma$  or  $\psi$ , then  $\psi \rightarrow \forall x \varphi(x)$  is a correct inference step.

If  $\varphi(a) \rightarrow \psi$  already occurs in the **derivation** and  $a$  does not occur in  $\Gamma$  or  $\psi$ , then  $\exists x \varphi(x) \rightarrow \psi$  is a correct inference step.

We'll abbreviate either of these by “QR.”

## axd.4 Examples of Derivations

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sec

**Example axd.9.** Suppose we want to prove  $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$ . Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to **derive** it. Our only rule is MP, which given  $\varphi$  and  $\varphi \rightarrow \psi$  allows us to justify  $\psi$ . One strategy would be to use [eq. \(axd.6\)](#) with  $\varphi$  being  $\neg\theta$ ,  $\psi$  being  $\alpha$ , and  $\chi$  being  $\theta \rightarrow \alpha$ , i.e., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that  $\neg\theta \rightarrow (\theta \rightarrow \alpha)$  is an instance of [eq. \(axd.10\)](#), and  $\alpha \rightarrow (\theta \rightarrow \alpha)$  is an instance of [eq. \(axd.7\)](#). So our derivation is:

1.  $\neg\theta \rightarrow (\theta \rightarrow \alpha)$  eq. (axd.7)
2.  $(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$   
 $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$  eq. (axd.6)
3.  $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$  1, 2, MP
4.  $\alpha \rightarrow (\theta \rightarrow \alpha)$  eq. (axd.7)
5.  $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$  3, 4, MP

**Example axd.10.** Let's try to find a **derivation** of  $\theta \rightarrow \theta$ . It is not an instance of an axiom, so we have to use MP to **derive** it. [eq. \(axd.7\)](#) is an axiom of the form  $\varphi \rightarrow \psi$  to which we could apply MP. To be useful, of course, the  $\psi$  which MP would justify as a correct step in this case would have to be  $\theta \rightarrow \theta$ , since this is what we want to **derive**. That means  $\varphi$  would also have to be  $\theta$ , i.e., we might look at this instance of [eq. \(axd.7\)](#):

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ex:identity

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely  $\varphi$ . But in our case, that would be  $\theta$ , and we won't be able to **derive**  $\theta$  by itself. So we need a different strategy.

The other axiom involving just  $\rightarrow$  is [eq. \(axd.8\)](#), i.e.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of [eq. \(axd.8\)](#) where  $\varphi \rightarrow \chi$  is  $\theta \rightarrow \theta$ , the **formula** we are aiming for. Then of course,  $\varphi$  and  $\chi$  are both  $\theta$ . How should we pick  $\psi$  so that both  $\varphi \rightarrow (\psi \rightarrow \chi)$  and  $\varphi \rightarrow \psi$ , i.e., in our case  $\theta \rightarrow (\psi \rightarrow \theta)$  and  $\theta \rightarrow \psi$ , are also **derivable**? Well, the first of these is already an instance of [eq. \(axd.7\)](#), whatever we decide  $\psi$  to be. And  $\theta \rightarrow \psi$  would be another instance of [eq. \(axd.7\)](#) if  $\psi$  were  $(\theta \rightarrow \theta)$ . So, our derivation is:

1.  $\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$  eq. (axd.7)
2.  $(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$   
 $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$  eq. (axd.8)
3.  $(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$  1, 2, MP
4.  $\theta \rightarrow (\theta \rightarrow \theta)$  eq. (axd.7)
5.  $\theta \rightarrow \theta$  3, 4, MP

**Example axd.11.** Sometimes we want to show that there is a derivation of some formula from some other formulas  $\Gamma$ . For instance, let's show that we can derive  $\varphi \rightarrow \chi$  from  $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$ .

1.  $\varphi \rightarrow \psi$  HYP
2.  $\psi \rightarrow \chi$  HYP
3.  $(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$  eq. (axd.7)
4.  $\varphi \rightarrow (\psi \rightarrow \chi)$  2, 3, MP
5.  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$   
 $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$  eq. (axd.8)
6.  $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$  4, 5, MP
7.  $\varphi \rightarrow \chi$  1, 6, MP

The lines labelled “HYP” (for “hypothesis”) indicate that the formula on that line is an element of  $\Gamma$ .

**Proposition axd.12.** *If  $\Gamma \vdash \varphi \rightarrow \psi$  and  $\Gamma \vdash \psi \rightarrow \chi$ , then  $\Gamma \vdash \varphi \rightarrow \chi$*

*Proof.* Suppose  $\Gamma \vdash \varphi \rightarrow \psi$  and  $\Gamma \vdash \psi \rightarrow \chi$ . Then there is a derivation of  $\varphi \rightarrow \psi$  from  $\Gamma$ ; and a derivation of  $\psi \rightarrow \chi$  from  $\Gamma$  as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of  $\varphi \rightarrow \chi$ —which is the last line of the new derivation—from  $\Gamma$ . Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of  $\varphi \rightarrow \psi$ , and reference to line number 1 by reference to the last line of the derivation of  $\psi \rightarrow \chi$ .  $\square$

**Problem axd.1.** Show that the following hold by exhibiting derivations from the axioms:

1.  $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2.  $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3.  $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

## axd.5 Derivations with Quantifiers

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**Example axd.13.** Let us give a derivation of  $(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x (\varphi(x) \wedge \psi(x))$ .

First, note that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x \varphi(x)$$

is an instance of eq. (axd.1), and

$$\forall x \varphi(x) \rightarrow \varphi(a)$$

of eq. (axd.15). So, by Proposition axd.12, we know that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \varphi(a)$$

is derivable. Likewise, since

$$\begin{aligned} (\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall y \psi(y) & \quad \text{and} \\ \forall y \psi(y) \rightarrow \psi(a) & \end{aligned}$$

are instances of eq. (axd.2) and eq. (axd.15), respectively,

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \psi(a)$$

is derivable by Proposition axd.12. Using an appropriate instance of eq. (axd.3) and two applications of MP, we see that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow (\varphi(a) \wedge \psi(a))$$

is derivable. We can now apply QR to obtain

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x (\varphi(x) \wedge \psi(x)).$$

## axd.6 Proof-Theoretic Notions

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. fol:axd:ptn:sec These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

**Definition axd.14 (Derivability).** A formula  $\varphi$  is *derivable* from  $\Gamma$ , written  $\Gamma \vdash \varphi$ , if there is a derivation from  $\Gamma$  ending in  $\varphi$ .

**Definition axd.15 (Theorems).** A formula  $\varphi$  is a *theorem* if there is a derivation of  $\varphi$  from the empty set. We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition axd.16** (Consistency). A set  $\Gamma$  of **formulas** is *consistent* if and only if  $\Gamma \not\vdash \perp$ ; it is *inconsistent* otherwise.

*fol:axd:ptn:  
prop:reflexivity*

**Proposition axd.17** (Reflexivity). *If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .*

*Proof.* The **formula**  $\varphi$  by itself is a **derivation** of  $\varphi$  from  $\Gamma$ . □

*fol:axd:ptn:  
prop:monotony*

**Proposition axd.18** (Monotony). *If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$ .*

*Proof.* Any **derivation** of  $\varphi$  from  $\Gamma$  is also a **derivation** of  $\varphi$  from  $\Delta$ . □

*fol:axd:ptn:  
prop:transitivity*

**Proposition axd.19** (Transitivity). *If  $\Gamma \vdash \varphi$  and  $\{\varphi\} \cup \Delta \vdash \psi$ , then  $\Gamma \cup \Delta \vdash \psi$ .*

*Proof.* Suppose  $\{\varphi\} \cup \Delta \vdash \psi$ . Then there is a **derivation**  $\psi_1, \dots, \psi_l = \psi$  from  $\{\varphi\} \cup \Delta$ . Some of the steps in that derivation will be correct because of a rule which refers to a prior line  $\psi_i = \varphi$ . By hypothesis, there is a **derivation** of  $\varphi$  from  $\Gamma$ , i.e., a **derivation**  $\varphi_1, \dots, \varphi_k = \varphi$  where every  $\varphi_i$  is an axiom, an **element** of  $\Gamma$ , or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct **derivation** of  $\psi$  from  $\Gamma \cup \Delta$  since every  $B_i = \varphi$  is now justified by the same rule which justifies  $\varphi_k = \varphi$ . □

Note that this means that in particular if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . It follows also that if  $\varphi_1, \dots, \varphi_n \vdash \psi$  and  $\Gamma \vdash \varphi_i$  for each  $i$ , then  $\Gamma \vdash \psi$ .

*fol:axd:ptn:  
prop:incons*

**Proposition axd.20.**  *$\Gamma$  is inconsistent iff  $\Gamma \vdash \varphi$  for every  $\varphi$ .*

*Proof.* Exercise. □

**Problem axd.2.** Prove **Proposition axd.20**.

*fol:axd:ptn:  
prop:proves-compact*

**Proposition axd.21** (Compactness).

1. *If  $\Gamma \vdash \varphi$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash \varphi$ .*
2. *If every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  is consistent.*

*Proof.* 1. If  $\Gamma \vdash \varphi$ , then there is a finite sequence of **formulas**  $\varphi_1, \dots, \varphi_n$  so that  $\varphi \equiv \varphi_n$  and each  $\varphi_i$  is either a logical axiom, an **element** of  $\Gamma$  or follows from previous **formulas** by modus ponens. Take  $\Gamma_0$  to be those  $\varphi_i$  which are in  $\Gamma$ . Then the **derivation** is likewise a **derivation** from  $\Gamma_0$ , and so  $\Gamma_0 \vdash \varphi$ .

2. This is the contrapositive of (1) for the special case  $\varphi \equiv \perp$ . □

## axd.7 The Deduction Theorem

As we've seen, giving **derivations** in an axiomatic system is cumbersome, and **derivations** may be hard to find. Rather than actually write out long lists of **formulas**, it is generally easier to argue that such **derivations** exist, by making use of a few simple results. We've already established three such results: [Proposition axd.17](#) says we can always assert that  $\Gamma \vdash \varphi$  when we know that  $\varphi \in \Gamma$ . [Proposition axd.18](#) says that if  $\Gamma \vdash \varphi$  then also  $\Gamma \cup \{\psi\} \vdash \varphi$ . And [Proposition axd.19](#) implies that if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . Here's another simple result, a "meta"-version of modus ponens:

**Proposition axd.22.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \varphi \rightarrow \psi$ , then  $\Gamma \vdash \psi$ .*

fol:axd:ded:  
sec

fol:axd:ded:  
prop:mp

*Proof.* We have that  $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$ :

1.  $\varphi$  Hyp.
2.  $\varphi \rightarrow \psi$  Hyp.
3.  $\psi$  1, 2, MP

By [Proposition axd.19](#),  $\Gamma \vdash \psi$ . □

The most important result we'll use in this context is the deduction theorem:

**Theorem axd.23** (Deduction Theorem).  *$\Gamma \cup \{\varphi\} \vdash \psi$  if and only if  $\Gamma \vdash \varphi \rightarrow \psi$ .*

fol:axd:ded:  
thm:deduction-thm

*Proof.* The "if" direction is immediate. If  $\Gamma \vdash \varphi \rightarrow \psi$  then also  $\Gamma \cup \{\varphi\} \vdash \varphi \rightarrow \psi$  by [Proposition axd.18](#). Also,  $\Gamma \cup \{\varphi\} \vdash \varphi$  by [Proposition axd.17](#). So, by [Proposition axd.22](#),  $\Gamma \cup \{\varphi\} \vdash \psi$ .

For the "only if" direction, we proceed by induction on the length of the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$ .

For the induction basis, we prove the claim for every **derivation** of length 1. A **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  of length 1 consists of  $\psi$  by itself; and if it is correct  $\psi$  is either  $\in \Gamma \cup \{\varphi\}$  or is an axiom. If  $\psi \in \Gamma$  or is an axiom, then  $\Gamma \vdash \psi$ . We also have that  $\Gamma \vdash \psi \rightarrow (\varphi \rightarrow \psi)$  by [eq. \(axd.7\)](#), and [Proposition axd.22](#) gives  $\Gamma \vdash \varphi \rightarrow \psi$ . If  $\psi \in \{\varphi\}$  then  $\Gamma \vdash \varphi \rightarrow \psi$  because then last **sentence**  $\varphi \rightarrow \psi$  is the same as  $\varphi \rightarrow \varphi$ , and we have **derived** that in [Example axd.10](#).

For the inductive step, suppose a **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  ends with a step  $\psi$  which is justified by modus ponens. (If it is not justified by modus ponens,  $\psi \in \Gamma$ ,  $\psi \equiv \varphi$ , or  $\psi$  is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the **derivation** are  $\chi \rightarrow \psi$  and  $\chi$ , for some **formula**  $\chi$ , i.e.,  $\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \psi$  and  $\Gamma \cup \{\varphi\} \vdash \chi$ , and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\begin{aligned} \Gamma \vdash \varphi \rightarrow (\chi \rightarrow \psi); \\ \Gamma \vdash \varphi \rightarrow \chi. \end{aligned}$$

But also

$$\Gamma \vdash (\varphi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)),$$



by eq. (axd.8), and two applications of Proposition axd.22 give  $\Gamma \vdash \varphi \rightarrow \psi$ , as required.  $\square$

Notice how eq. (axd.7) and eq. (axd.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about **derivability**, which we leave as exercises.

*fol:axd:ded:prop:derivfacts* **Proposition axd.24.**

- fol:axd:ded:derivfacts:a* 1.  $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$ ;
- fol:axd:ded:derivfacts:b* 2. If  $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$  then  $\Gamma \cup \{\psi\} \vdash \varphi$  (Contraposition);
- fol:axd:ded:derivfacts:c* 3.  $\{\varphi, \neg\varphi\} \vdash \psi$  (Ex Falso Quodlibet, Explosion);
- fol:axd:ded:derivfacts:d* 4.  $\{\neg\neg\varphi\} \vdash \varphi$  (Double Negation Elimination);
- fol:axd:ded:derivfacts:e* 5. If  $\Gamma \vdash \neg\neg\varphi$  then  $\Gamma \vdash \varphi$ ;

**Problem axd.3.** Prove Proposition axd.24

## axd.8 The Deduction Theorem with Quantifiers

*fol:axd:ddq:sec*

*fol:axd:ddq:thm:deduction-thm-q* **Theorem axd.25** (Deduction Theorem). *If  $\Gamma \cup \{\varphi\} \vdash \psi$  and then  $\Gamma \vdash \varphi \rightarrow \psi$ .*

*Proof.* We again proceed by induction on the length of the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$ .

The proof of the induction basis is identical to that in the proof of Theorem axd.23.

For the inductive step, suppose again that the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  ends with a step  $\psi$  which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of Theorem axd.23. If the inference rule is QR, we know that  $\psi \equiv \chi \rightarrow \forall x \theta(x)$  and a **formula** of the form  $\chi \rightarrow \theta(a)$  appears earlier in the **derivation**, where  $a$  does not occur in  $\chi$ ,  $\varphi$ , or  $\Gamma$ . We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \theta(a)$$

and the induction hypothesis applies, i.e., we have that

$$\Gamma \vdash \varphi \rightarrow \theta(a)$$

By

$$\vdash (\varphi \rightarrow (\chi \rightarrow \theta(a))) \rightarrow ((\varphi \wedge \chi) \rightarrow \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \theta(a).$$

Since the eigenvariable condition still applies, we can add a step to this [derivation](#) justified by QR, and get:

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \forall x \theta(x)$$

We also have

$$\vdash ((\varphi \wedge \chi) \rightarrow \forall x \theta(x)) \rightarrow (\varphi \rightarrow (\chi \rightarrow \forall x \theta(x)))$$

so by modus ponens,

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \forall x \theta(x))$$

i.e.,  $\Gamma \vdash \psi$ .

We leave the case where  $\psi$  is justified by the rule QR, but is of the form  $\exists x \theta(x) \rightarrow \chi$ , as an exercise.  $\square$

**Problem axd.4.** Complete the proof of [Theorem axd.25](#).

## axd.9 Derivability and Consistency

We will now establish a number of properties of the [derivability](#) relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

[fol:axd:prv:](#)  
[sec](#)

**Proposition axd.26.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

[fol:axd:prv:](#)  
[prop:provability-contr](#)

*Proof.* If  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma \cup \{\varphi\} \vdash \perp$ . By [Proposition axd.17](#),  $\Gamma \vdash \psi$  for every  $\psi \in \Gamma$ . Since also  $\Gamma \vdash \varphi$  by hypothesis,  $\Gamma \vdash \psi$  for every  $\psi \in \Gamma \cup \{\varphi\}$ . By [Proposition axd.19](#),  $\Gamma \vdash \perp$ , i.e.,  $\Gamma$  is inconsistent.  $\square$

**Proposition axd.27.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.*

[fol:axd:prv:](#)  
[prop:prov-incons](#)

*Proof.* First suppose  $\Gamma \vdash \varphi$ . Then  $\Gamma \cup \{\neg\varphi\} \vdash \varphi$  by [Proposition axd.18](#).  $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$  by [Proposition axd.17](#). We also have  $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$  by [eq. \(axd.10\)](#). So by two applications of [Proposition axd.22](#), we have  $\Gamma \cup \{\neg\varphi\} \vdash \perp$ .

Now assume  $\Gamma \cup \{\neg\varphi\}$  is inconsistent, i.e.,  $\Gamma \cup \{\neg\varphi\} \vdash \perp$ . By the deduction theorem,  $\Gamma \vdash \neg\varphi \rightarrow \perp$ .  $\Gamma \vdash (\neg\varphi \rightarrow \perp) \rightarrow \neg\neg\varphi$  by [eq. \(axd.13\)](#), so  $\Gamma \vdash \neg\neg\varphi$  by [Proposition axd.22](#). Since  $\Gamma \vdash \neg\neg\varphi \rightarrow \varphi$  ([eq. \(axd.14\)](#)), we have  $\Gamma \vdash \varphi$  by [Proposition axd.22](#) again.  $\square$

**Problem axd.5.** Prove that  $\Gamma \vdash \neg\varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

*fol:axd:prv:  
prop:explicit-inc*

**Proposition axd.28.** *If  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.*  $\Gamma \vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$  by eq. (axd.10).  $\Gamma \vdash \perp$  by two applications of Proposition axd.22.  $\square$

*fol:axd:prv:  
prop:provability-exhaustive*

**Proposition axd.29.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* Exercise.  $\square$

**Problem axd.6.** Prove Proposition axd.29

## axd.10 Derivability and the Propositional Connectives

*fol:axd:ppr:  
sec*

*fol:axd:ppr:  
prop:provability-land*

**Proposition axd.30.**

*fol:axd:ppr:  
prop:provability-land-left*

1. Both  $\varphi \wedge \psi \vdash \varphi$  and  $\varphi \wedge \psi \vdash \psi$

*fol:axd:ppr:  
prop:provability-land-right*

2.  $\varphi, \psi \vdash \varphi \wedge \psi$ .

*Proof.* 1. From eq. (axd.1) and eq. (axd.1) by modus ponens.

2. From eq. (axd.3) by two applications of modus ponens.  $\square$

*fol:axd:ppr:  
prop:provability-lor*

**Proposition axd.31.**

1.  $\varphi \vee \psi, \neg\varphi, \neg\psi$  is inconsistent.

2. Both  $\varphi \vdash \varphi \vee \psi$  and  $\psi \vdash \varphi \vee \psi$ .

*Proof.* 1. From eq. (axd.9) we get  $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$  and  $\vdash \neg\psi \rightarrow (\psi \rightarrow \perp)$ . So by the deduction theorem, we have  $\{\neg\varphi\} \vdash \varphi \rightarrow \perp$  and  $\{\neg\psi\} \vdash \psi \rightarrow \perp$ . From eq. (axd.6) we get  $\{\neg\varphi, \neg\psi\} \vdash (\varphi \vee \psi) \rightarrow \perp$ . By the deduction theorem,  $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \vdash \perp$ .

2. From eq. (axd.4) and eq. (axd.5) by modus ponens.  $\square$

*fol:axd:ppr:  
prop:provability-lif*

**Proposition axd.32.**

*fol:axd:ppr:  
prop:provability-lif-left*

1.  $\varphi, \varphi \rightarrow \psi \vdash \psi$ .

*fol:axd:ppr:  
prop:provability-lif-right*

2. Both  $\neg\varphi \vdash \varphi \rightarrow \psi$  and  $\psi \vdash \varphi \rightarrow \psi$ .

*Proof.* 1. We can derive:

1.  $\varphi$             HYP
2.  $\varphi \rightarrow \psi$     HYP
3.  $\psi$              1, 2, MP

2. By eq. (axd.10) and eq. (axd.7) and the deduction theorem, respectively.  $\square$

## axd.11 Derivability and the Quantifiers

fol:axd:qpr:  
sec  
fol:axd:qpr:  
thm:strong-generalization

**Theorem axd.33.** *If  $c$  is a constant symbol not occurring in  $\Gamma$  or  $\varphi(x)$  and  $\Gamma \vdash \varphi(c)$ , then  $\Gamma \vdash \forall x \varphi(x)$ .*

*Proof.* By the deduction theorem,  $\Gamma \vdash \top \rightarrow \varphi(c)$ . Since  $c$  does not occur in  $\Gamma$  or  $\top$ , we get  $\Gamma \vdash \top \rightarrow \varphi(c)$ . By the deduction theorem again,  $\Gamma \vdash \forall x \varphi(x)$ .  $\square$

**Proposition axd.34.**

fol:axd:qpr:  
prop:provability-quantifiers

1.  $\varphi(t) \vdash \exists x \varphi(x)$ .
2.  $\forall x \varphi(x) \vdash \varphi(t)$ .

*Proof.* 1. By eq. (axd.16) and the deduction theorem.  
2. By eq. (axd.15) and the deduction theorem.  $\square$

## axd.12 Soundness

explanation A derivation system, such as axiomatic deduction, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that fol:axd:sou:  
sec

1. every derivable  $\varphi$  is valid;
2. if  $\varphi$  is derivable from some others  $\Gamma$ , it is also a consequence of them;
3. if a set of formulas  $\Gamma$  is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

**Proposition axd.35.** *If  $\varphi$  is an axiom, then  $\mathfrak{M}, s \models \varphi$  for each structure  $\mathfrak{M}$  and assignment  $s$ .*

*Proof.* We have to verify that all the axioms are valid. For instance, here is the case for eq. (axd.15): suppose  $t$  is **free for**  $x$  in  $\varphi$ , and assume  $\mathfrak{M}, s \models \forall x \varphi$ . Then by definition of satisfaction, for each  $s' \sim_x s$ , also  $\mathfrak{M}, s' \models \varphi$ , and in particular this holds when  $s'(x) = \text{Val}_s^{\mathfrak{M}}(t)$ . By ??,  $\mathfrak{M}, s \models \varphi[t/x]$ . This shows that  $\mathfrak{M}, s \models (\forall x \varphi \rightarrow \varphi[t/x])$ .  $\square$

*fol:axd:sou:*  
*thm:soundness*

**Theorem axd.36** (Soundness). *If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .*

*Proof.* By induction on the length of the **derivation** of  $\varphi$  from  $\Gamma$ . If there are no steps justified by inferences, then all **formulas** in the derivation are either instances of axioms or are in  $\Gamma$ . By the previous proposition, all the axioms are valid, and hence if  $\varphi$  is an axiom then  $\Gamma \models \varphi$ . If  $\varphi \in \Gamma$ , then trivially  $\Gamma \models \varphi$ .

If the last step of the derivation of  $\varphi$  is justified by modus ponens, then there are **formulas**  $\psi$  and  $\psi \rightarrow \varphi$  in the **derivation**, and the induction hypothesis applies to the part of the **derivation** ending in those **formulas** (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis,  $\Gamma \models \psi$  and  $\Gamma \models \psi \rightarrow \varphi$ . Then  $\Gamma \models \varphi$  by ??.

Now suppose the last step is justified by QR. Then that step has the form  $\chi \rightarrow \forall x B(x)$  and there is a preceding step  $\chi \rightarrow \psi(c)$  with  $c$  not in  $\Gamma$ ,  $\chi$ , or  $\forall x B(x)$ . By induction hypothesis,  $\Gamma \models \chi \rightarrow \forall x B(x)$ . By ??,  $\Gamma \cup \{\chi\} \models \psi(c)$ .

Consider some structure  $\mathfrak{M}$  such that  $\mathfrak{M} \models \Gamma \cup \{\chi\}$ . We need to show that  $\mathfrak{M} \models \forall x \psi(x)$ . Since  $\forall x \psi(x)$  is a **sentence**, this means we have to show that for every variable assignment  $s$ ,  $\mathfrak{M}, s \models \psi(x)$  (??). Since  $\Gamma \cup \{\chi\}$  consists entirely of sentences,  $\mathfrak{M}, s \models \theta$  for all  $\theta \in \Gamma$  by ??. Let  $\mathfrak{M}'$  be like  $\mathfrak{M}$  except that  $c^{\mathfrak{M}'} = s(x)$ . Since  $c$  does not occur in  $\Gamma$  or  $\chi$ ,  $\mathfrak{M}' \models \Gamma \cup \{\chi\}$  by ??. Since  $\Gamma \cup \{\chi\} \models \psi(c)$ ,  $\mathfrak{M}' \models B(c)$ . Since  $\psi(c)$  is a **sentence**,  $\mathfrak{M}, s \models \psi(c)$  by ??.  $\mathfrak{M}', s \models \psi(x)$  iff  $\mathfrak{M}' \models \psi(c)$  by ?? (recall that  $\psi(c)$  is just  $\psi(x)[c/x]$ ). So,  $\mathfrak{M}', s \models \psi(x)$ . Since  $c$  does not occur in  $\psi(x)$ , by ??,  $\mathfrak{M}, s \models \psi(x)$ . But  $s$  was an arbitrary variable assignment, so  $\mathfrak{M} \models \forall x \psi(x)$ . Thus  $\Gamma \cup \{\chi\} \models \forall x \psi(x)$ . By ??,  $\Gamma \models \chi \rightarrow \forall x \psi(x)$ .

The case where  $\varphi$  is justified by QR but is of the form  $\exists x \psi(x) \rightarrow \chi$  is left as an exercise.  $\square$

**Problem axd.7.** Complete the proof of [Theorem axd.36](#).

*fol:axd:sou:*  
*cor:weak-soundness*

**Corollary axd.37.** *If  $\vdash \varphi$ , then  $\varphi$  is valid.*

*fol:axd:sou:*  
*cor:consistency-soundness*

**Corollary axd.38.** *If  $\Gamma$  is satisfiable, then it is consistent.*

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then  $\Gamma \vdash \perp$ , i.e., there is a **derivation** of  $\perp$  from  $\Gamma$ . By [Theorem axd.36](#), any **structure**  $\mathfrak{M}$  that satisfies  $\Gamma$  must satisfy  $\perp$ . Since  $\mathfrak{M} \not\models \perp$  for every **structure**  $\mathfrak{M}$ , no  $\mathfrak{M}$  can satisfy  $\Gamma$ , i.e.,  $\Gamma$  is not satisfiable.  $\square$

### axd.13 Derivations with Identity predicate

In order to accommodate  $=$  in **derivations**, we simply add new axiom schemas. The definition of **derivation** and  $\vdash$  remains the same, we just also allow the new axioms. fol:axd:ide:sec

**Definition axd.39** (Axioms for **identity predicate**).

$$t = t, \tag{axd.17} \small \text{fol:axd:ide:}$$

$$t_1 = t_2 \rightarrow (\psi(t_1) \rightarrow \psi(t_2)), \tag{axd.18} \small \text{ax:id1:}$$

fol:axd:ide:ax:id2

for any ground terms  $t, t_1, t_2$ .

**Proposition axd.40.** *The axioms eq. (axd.17) and eq. (axd.18) are valid.* fol:axd:ide:prop:sound

*Proof.* Exercise. □

**Problem axd.8.** Prove **Proposition axd.40**.

**Proposition axd.41.**  $\Gamma \vdash t = t$ , for any term  $t$  and set  $\Gamma$ .

fol:axd:ide:prop:iden1

**Proposition axd.42.** If  $\Gamma \vdash \varphi(t_1)$  and  $\Gamma \vdash t_1 = t_2$ , then  $\Gamma \vdash \varphi(t_2)$ .

fol:axd:ide:prop:iden2

*Proof.* The **formula**

$$(t_1 = t_2 \rightarrow (\varphi(t_1) \rightarrow \varphi(t_2)))$$

is an instance of eq. (axd.18). The conclusion follows by two applications of MP. □

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# Bibliography