Chapter udf

Axiomatic Derivations

axd.1 Rules and Derivations

fol:axd:rul:sec Axiomatic derivations are perhaps the simplest proof system for logic. A derivation is just a sequence of formulas. To count as a derivation, every formula in the sequence must either be an instance of an axiom, or must follow from one or more formulas that precede it in the sequence by a rule of inference. A derivation derives its last formula. explanation

Definition axd.1 (Derivability)**.** If Γ is a set of formulas of $\mathcal{L}$ then a *derivation* from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct derivation depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in is a correct inference step.

Definition axd.2 (Rule of inference)**.** A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a derivation from Γ.

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a derivation, the following might be a very simple rule of inference:

> If $\varphi \in \Gamma$, then φ is always a correct inference step in any derivation from Γ.

Similarly, if φ is one of the axioms, then φ by itself is a derivation, and so this is also a rule of inference:

If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to formulas that appear before the step considered. The following rule is called *modus ponens:*

If $\psi \to \varphi$ and ψ occur higher up in the derivation, then φ is a correct inference step.

If this is the only rule of inference, then our definition of derivation above amounts to this: $\varphi_1, \dots, \varphi_n$ is a derivation iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. for some $j < i$, φ_j is $\psi \to \varphi_i$, and for some $k < i$, φ_k is ψ.

The last clause says that φ_i follows from φ_j (ψ) and φ_k ($\psi \to \varphi_i$) by modus ponens. If we can go from 1 to n, and each time we find a formula φ_i that is either in Γ, an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivation.

Definition axd.3 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition axd.4 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

axd.2 Axiom and Rules for the Propositional Connectives

fol:axd:prp:sec

Definition axd.5 (Axioms)**.** The set of Ax_0 of *axioms* for the propositional connectives comprises all formulas of the following forms:

fol:axd:prp:ax:land1
$$(\varphi \land \psi) \to \varphi \tag{axd.1}$$
fol:axd:prp:ax:land2
$$(\varphi \land \psi) \to \psi \tag{axd.2}$$
fol:axd:prp:ax:land3
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{axd.3}$$
fol:axd:prp:ax:lor1
$$\varphi \to (\varphi \lor \psi) \tag{axd.4}$$
fol:axd:prp:ax:lor2
$$\varphi \to (\psi \lor \varphi) \tag{axd.5}$$
fol:axd:prp:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{axd.6}$$
fol:axd:prp:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{axd.7}$$
fol:axd:prp:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{axd.8}$$
fol:axd:prp:ax:lnot1
$$(\varphi \to \psi) \to ((\varphi \to \lnot\psi) \to \lnot\varphi) \tag{axd.9}$$
fol:axd:prp:ax:lnot2
$$\lnot\varphi \to (\varphi \to \psi) \tag{axd.10}$$
fol:axd:prp:ax:ltrue
$$\top \tag{axd.11}$$
fol:axd:prp:ax:lfalse1
$$\bot \to \varphi \tag{axd.12}$$
fol:axd:prp:ax:lfalse2
$$(\varphi \to \bot) \to \lnot\varphi \tag{axd.13}$$
fol:axd:prp:ax:dne
$$\lnot\lnot\varphi \to \varphi \tag{axd.14}$$

Definition axd.6 (Modus ponens)**.** If ψ and $\psi \to \varphi$ already occur in a derivation, then φ is a correct inference step.

We'll abbreviate the rule modus ponens as "MP."

axd.3 Axioms and Rules for Quantifiers

fol:axd:qua:sec

Definition axd.7 (Axioms for quantifiers)**.** The *axioms* governing quantifiers are all instances of the following:

fol:axd:qua:ax:q1
$$\forall x\, \psi \to \psi(t), \tag{axd.15}$$
fol:axd:qua:ax:q2
$$\psi(t) \to \exists x\, \psi. \tag{axd.16}$$

for any ground term t.

Definition axd.8 (Rules for quantifiers)**.**

If $\psi \to \varphi(a)$ already occurs in the derivation and a does not occur in Γ or ψ, then $\psi \to \forall x\, \varphi(x)$ is a correct inference step.

If $\varphi(a) \to \psi$ already occurs in the derivation and a does not occur in Γ or ψ, then $\exists x\, \varphi(x) \to \psi$ is a correct inference step.

We'll abbreviate either of these by "QR."

axd.4 Examples of Derivations

fol:axd:pro:sec

Example axd.9. Suppose we want to prove $(\lnot\theta \lor \alpha) \to (\theta \to \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to derive it. Our only rule is MP, which given φ and $\varphi \to \psi$ allows us to justify ψ. One strategy would be to use eq. (axd.6) with φ being $\lnot\theta$, ψ being α, and χ being $\theta \to \alpha$, i.e., the instance

$$(\lnot\theta \to (\theta \to \alpha)) \to ((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\lnot\theta \to (\theta \to \alpha)$ is an instance of eq. (axd.10), and $\alpha \to (\theta \to \alpha)$ is an instance of eq. (axd.7). So our derivation is:

1.	$\lnot\theta \to (\theta \to \alpha)$	eq. (axd.7)
2.	$(\lnot\theta \to (\theta \to \alpha)) \to$ $((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha)))$	eq. (axd.6)
3.	$((\alpha \to (\theta \to \alpha)) \to ((\lnot\theta \lor \alpha) \to (\theta \to \alpha))$	1, 2, MP
4.	$\alpha \to (\theta \to \alpha)$	eq. (axd.7)
5.	$(\lnot\theta \lor \alpha) \to (\theta \to \alpha)$	3, 4, MP

Example axd.10. Let's try to find a derivation of $\theta \to \theta$. It is not an instance of an axiom, so we have to use MP to derive it. eq. (axd.7) is an axiom of the form $\varphi \to \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \to \theta$, since this is what we want to derive. That means φ would also have to be θ, i.e., we might look at this instance of eq. (axd.7): fol:axd:pro:ex:identity

$$\theta \to (\theta \to \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ. But in our case, that would be θ, and we won't be able to derive θ by itself. So we need a different strategy.

The other axiom involving just $\to$ is eq. (axd.8), i.e.,

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of eq. (axd.8) where $\varphi \to \chi$ is $\theta \to \theta$, the formula we are aiming for. Then of course, φ and χ are both θ. How should we pick ψ so that both $\varphi \to (\psi \to \chi)$ and $\varphi \to \psi$, i.e., in our case $\theta \to (\psi \to \theta)$ and $\theta \to \psi$, are also derivable? Well, the first of these is already an instance of eq. (axd.7), whatever we decide ψ to be. And $\theta \to \psi$ would be another instance of eq. (axd.7) if ψ were $(\theta \to \theta)$. So, our derivation is:

1.	$\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$	eq. (axd.7)
2.	$(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$ $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$	eq. (axd.8)
3.	$(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$	1, 2, MP
4.	$\theta \rightarrow (\theta \rightarrow \theta)$	eq. (axd.7)
5.	$\theta \rightarrow \theta$	3, 4, MP

fol:axd:pro:ex:chain **Example axd.11.** Sometimes we want to show that there is a derivation of some formula from some other formulas Γ. For instance, let's show that we can derive $\varphi \rightarrow \chi$ from $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$.

1.	$\varphi \rightarrow \psi$	Hyp
2.	$\psi \rightarrow \chi$	Hyp
3.	$(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$	eq. (axd.7)
4.	$\varphi \rightarrow (\psi \rightarrow \chi)$	2, 3, MP
5.	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$ $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	eq. (axd.8)
6.	$((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	4, 5, MP
7.	$\varphi \rightarrow \chi$	1, 6, MP

The lines labelled "Hyp" (for "hypothesis") indicate that the formula on that line is an element of Γ.

fol:axd:pro:prop:chain **Proposition axd.12.** *If $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$, then $\Gamma \vdash \varphi \rightarrow \chi$*

Proof. Suppose $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$. Then there is a derivation of $\varphi \rightarrow \psi$ from Γ; and a derivation of $\psi \rightarrow \chi$ from Γ as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of $\varphi \rightarrow \chi$—which is the last line of the new derivation—from Γ. Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of $\varphi \rightarrow \psi$, and reference to line number 1 by reference to the last line of the derivation of $B \rightarrow \chi$. □

Problem axd.1. Show that the following hold by exhibiting derivations from the axioms:

1. $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3. $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

axd.5 Derivations with Quantifiers

fol:axd:prq:sec

Example axd.13. Let us give a derivation of $(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, (\varphi(x) \land \psi(x))$.

First, note that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, \varphi(x)$$

is an instance of eq. (axd.1), and

$$\forall x\, \varphi(x) \to \varphi(a)$$

of eq. (axd.15). So, by Proposition axd.12, we know that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \varphi(a)$$

is derivable. Likewise, since

$$\begin{aligned} (\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall y\, \psi(y) \qquad & \text{and} \\ \forall y\, \psi(y) \to \psi(a) \qquad & \end{aligned}$$

are instances of eq. (axd.2) and eq. (axd.15), respectively,

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \psi(a)$$

is derivable by Proposition axd.12. Using an appropriate instance of eq. (axd.3) and two applications of MP, we see that

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to (\varphi(a) \land \psi(a))$$

is derivable. We can now apply QR to obtain

$$(\forall x\, \varphi(x) \land \forall y\, \psi(y)) \to \forall x\, (\varphi(x) \land \psi(x)).$$

axd.6 Proof-Theoretic Notions

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain formulas. It was an important discovery, due to Gödel, that these notions coincide. That they do is the content of the *completeness theorem*. fol:axd:ptn:sec

The proof-theoretic notions for propositional logic are similar.

Definition axd.14 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition axd.15 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition axd.16 (Consistency)**.** A set Γ of formulas is *consistent* if and only if $\Gamma \nvdash \bot$; it is *inconsistent* otherwise.

fol:axd:ptn:prop:reflexivity **Proposition axd.17** (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The formula φ by itself is a derivation of φ from Γ. □

fol:axd:ptn:prop:monotony **Proposition axd.18** (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any derivation of φ from Γ is also a derivation of φ from Δ. □

fol:axd:ptn:prop:transitivity **Proposition axd.19** (Transitivity)**.** *If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.*

Proof. Suppose $\Delta \vdash \psi$. Then there is a derivation $\psi_1, \ldots, \psi_l = \psi$ from Δ. Some of the steps in that derivation will be correct because of a rule which refers to a prior line ψ_i that includes a formula $\psi_i \in \Delta$. Let $\varphi_1, \ldots, \varphi_n$ be all the formulas in Δ so referenced. For each one of them, by hypothesis, there is a derivation of φ_i from Γ, i.e., a derivation $\varphi_i^1, \ldots, \varphi_i^{k_i} = \varphi_i$ where every φ_i^j is an axiom, an element of Γ, or correct by a rule of inference. Now consider the sequence

$$\varphi_1^1, \ldots, \varphi_1^{k_1}, \ldots, \varphi_n^1, \ldots, \varphi_n^{k_n}, \psi_1, \ldots, \psi_l = \psi.$$

This is a correct derivation of ψ from Γ since each $\psi_i \in \Delta$ needed to justify the inferences in the part $\psi_1, \ldots, \psi_l$ is now justified itself. □

fol:axd:ptn:prop:incons **Proposition axd.20.** *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.*

Proof. Exercise. □

Problem axd.2. Prove Proposition axd.20

fol:axd:ptn:prop:proves-compact **Proposition axd.21** (Compactness)**.**

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof.
1. If $\Gamma \vdash \varphi$, then there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, an element of Γ or follows from previous formulas by modus ponens. Take Γ_0 to be those φ_i which are in Γ. Then the derivation is likewise a derivation from Γ_0, and so $\Gamma_0 \vdash \varphi$.
2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$.

□

axd.7 The Deduction Theorem

As we've seen, giving derivations in an axiomatic system is cumbersome, and derivations may be hard to find. Rather than actually write out long lists of formulas, it is generally easier to argue that such derivations exist, by making use of a few simple results. We've already established three such results: Proposition axd.17 says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. Proposition axd.18 says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And Proposition axd.19 implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens: fol:axd:ded:sec

Proposition axd.22. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.* *fol:axd:ded:prop:mp*

Proof. We have that $\{\varphi, \varphi \to \psi\} \vdash \psi$:

1.	φ	Hyp.
2.	$\varphi \to \psi$	Hyp.
3.	ψ	1, 2, MP

By Proposition axd.19, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

Theorem axd.23 (Deduction Theorem)**.** *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \to \psi$.* *fol:axd:ded:thm:deduction-thm*

Proof. The "if" direction is immediate. If $\Gamma \vdash \varphi \to \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$ by Proposition axd.18. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by Proposition axd.17. So, by Proposition axd.22, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the derivation of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every derivation of length 1. A derivation of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \to (\varphi \to \psi)$ by eq. (axd.7), and Proposition axd.22 gives $\Gamma \vdash \varphi \to \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \to \psi$ because then last sentence $\varphi \to \psi$ is the same as $\varphi \to \varphi$, and we have derived that in Example axd.10.

For the inductive step, suppose a derivation of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the derivation are $\chi \to \psi$ and χ, for some formula χ, i.e., $\Gamma \cup \{\varphi\} \vdash \chi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\Gamma \vdash \varphi \to (\chi \to \psi);$$
$$\Gamma \vdash \varphi \to \chi.$$

But also

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi)),$$

by eq. (axd.8), and two applications of Proposition axd.22 give $\Gamma \vdash \varphi \to \psi$, as required. □

Notice how eq. (axd.7) and eq. (axd.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about derivability, which we leave as exercises.

fol:axd:ded: prop:derivfacts

Proposition axd.24.

fol:axd:ded: derivfacts:a

1. $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)$*;*

fol:axd:ded: derivfacts:b

2. *If* $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ *then* $\Gamma \cup \{\psi\} \vdash \varphi$ *(Contraposition);*

fol:axd:ded: derivfacts:c

3. $\{\varphi, \neg\varphi\} \vdash \psi$ *(Ex Falso Quodlibet, Explosion);*

fol:axd:ded: derivfacts:d

4. $\{\neg\neg\varphi\} \vdash \varphi$ *(Double Negation Elimination);*

fol:axd:ded: derivfacts:e

5. *If* $\Gamma \vdash \neg\neg\varphi$ *then* $\Gamma \vdash \varphi$*;*

Problem axd.3. Prove Proposition axd.24

axd.8 The Deduction Theorem with Quantifiers

fol:axd:ddq: sec

fol:axd:ddq: thm:deduction-thm-q

Theorem axd.25 (Deduction Theorem)**.** *If* $\Gamma \cup \{\varphi\} \vdash \psi$ *and then* $\Gamma \vdash \varphi \to \psi$*.*

Proof. We again proceed by induction on the length of the derivation of ψ from $\Gamma \cup \{\varphi\}$.

The proof of the induction basis is identical to that in the proof of Theorem axd.23.

For the inductive step, suppose again that the derivation of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of Theorem axd.23. If the inference rule is QR, we know that $\psi \equiv \chi \to \forall x\, \theta(x)$ and a formula of the form $\chi \to \theta(a)$ appears earlier in the derivation, where a does not occur in χ, φ, or Γ. We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \to \theta(a)$$

and the induction hypothesis applies, i.e., we have that

$$\Gamma \vdash \varphi \to \theta(a)$$

By

$$\vdash (\varphi \to (\chi \to \theta(a))) \to ((\varphi \land \chi) \to \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \land \chi) \to \theta(a).$$

Since the eigenvariale condition still applies, we can add a step to this derivation justified by QR, and get:

$$\Gamma \vdash (\varphi \land \chi) \to \forall x\, \theta(x)$$

We also have

$$\vdash ((\varphi \land \chi) \to \forall x\, \theta(x)) \to (\varphi \to (\chi \to \forall x\, \theta(x))$$

so by modus ponens,

$$\Gamma \vdash \varphi \to (\chi \to \forall x\, \theta(x))$$

i.e., $\Gamma \vdash \psi$.

We leave the case where ψ is justified by the rule QR, but is of the form $\exists x\, \theta(x) \to \chi$, as an exercise. □

Problem axd.4. Complete the proof of Theorem axd.25.

axd.9 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. fol:axd:prv:sec

Proposition axd.26. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.* fol:axd:prv:prop:provability-contr

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By Proposition axd.17, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By Proposition axd.19, $\Gamma \vdash \bot$, i.e., Γ is inconsistent. □

Proposition axd.27. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* fol:axd:prv:prop:prov-incons

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition axd.18. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by Proposition axd.17. We also have $\vdash \neg\varphi \to (\varphi \to \bot)$ by eq. (axd.10). So by two applications of Proposition axd.22, we have $\Gamma \cup \{\neg\varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg\varphi \to \bot$. $\Gamma \vdash (\neg\varphi \to \bot) \to \neg\neg\varphi$ by eq. (axd.13), so $\Gamma \vdash \neg\neg\varphi$ by Proposition axd.22. Since $\Gamma \vdash \neg\neg\varphi \to \varphi$ (eq. (axd.14)), we have $\Gamma \vdash \varphi$ by Proposition axd.22 again. □

Problem axd.5. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:axd:prv: prop:explicit-inc

Proposition axd.28. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. $\Gamma \vdash \neg\varphi \to (\varphi \to \bot)$ by eq. (axd.10). $\Gamma \vdash \bot$ by two applications of Proposition axd.22. □

fol:axd:prv: prop:provability-exhaustive

Proposition axd.29. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. Exercise. □

Problem axd.6. Prove Proposition axd.29

axd.10 Derivability and the Propositional Connectives

fol:axd:ppr: sec

fol:axd:ppr: prop:provability-land

Proposition axd.30.

fol:axd:ppr: prop:provability-land-left

1. *Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$*

fol:axd:ppr: prop:provability-land-right

2. *$\varphi, \psi \vdash \varphi \wedge \psi$.*

Proof. 1. From eq. (axd.1) and eq. (axd.1) by modus ponens.

2. From eq. (axd.3) by two applications of modus ponens.

□

fol:axd:ppr: prop:provability-lor

Proposition axd.31.

1. *$\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.*

2. *Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.*

Proof. 1. From eq. (axd.9) we get $\vdash \neg\varphi \to (\varphi \to \bot)$ and $\vdash \neg\varphi \to (\varphi \to \bot)$. So by the deduction theorem, we have $\{\neg\varphi\} \vdash \varphi \to \bot$ and $\{\neg\psi\} \vdash \psi \to \bot$. From eq. (axd.6) we get $\{\neg\varphi, \neg\psi\} \vdash (\varphi \vee \psi) \to \bot$. By the deduction theorem, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \vdash \bot$.

2. From eq. (axd.4) and eq. (axd.5) by modus ponsens.

□

fol:axd:ppr: prop:provability-lif

Proposition axd.32.

fol:axd:ppr: prop:provability-lif-left

1. *$\varphi, \varphi \to \psi \vdash \psi$.*

fol:axd:ppr: prop:provability-lif-right

2. *Both $\neg\varphi \vdash \varphi \to \psi$ and $\psi \vdash \varphi \to \psi$.*

Proof. 1. We can derive:

1.	φ	Hyp
2.	$\varphi \to \psi$	Hyp
3.	ψ	1, 2, mp

2. By eq. (axd.10) and eq. (axd.7) and the deduction theorem, respectively.

□

axd.11 Derivability and the Quantifiers

fol:axd:qpr: sec

Theorem axd.33. *If c is a constant symbol not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x\, \varphi(x)$.* *fol:axd:qpr: thm:strong-generalization*

Proof. By the deduction theorem, $\Gamma \vdash \top \to \varphi(c)$. Since c does not occur in Γ or $\top$, we get $\Gamma \vdash \top \to \varphi(c)$. By the deduction theorem again, $\Gamma \vdash \forall x\, \varphi(x)$. □

Proposition axd.34. *fol:axd:qpr: prop:provability-quantifiers*

1. *$\varphi(t) \vdash \exists x\, \varphi(x)$.*
2. *$\forall x\, \varphi(x) \vdash \varphi(t)$.*

Proof. 1. By eq. (axd.16) and the deduction theorem.

2. By eq. (axd.15) and the deduction theorem.

□

axd.12 Soundness

explanation A derivation system, such as axiomatic deduction, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that fol:axd:sou: sec

1. every derivable sentence is valid;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Proposition axd.35. *If φ is an axiom, then $\mathfrak{M}, s \models \varphi$ for each structure $\mathfrak{M}$ and assignment s.*

Proof. We first verify that all the axioms are valid. For instance, here is the case for eq. (axd.15): suppose t is free for x in φ, and assume $\mathfrak{M}, s \models \forall x\, \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \models \varphi$, and in particular this holds when $s'(x) = \mathrm{Val}^{\mathfrak{M}}_s(t)$. By **??**, $\mathfrak{M}, s \models \varphi[t/x]$. This shows that $\mathfrak{M}, s \models (\forall x\, \varphi \to \varphi[t/x])$. □

fol:axd:sou: thm:soundness

Theorem axd.36 (Soundness). *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Proof. By induction on the length of the derivation of φ from Γ. If there are no steps justified by inferences, then all formulas in the derivation are either instances of axioms or are in Γ. By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \vDash \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \vDash \varphi$.

If the last step of the derivation of φ is justified by modus ponens, then there are formulas ψ and $\psi \to \varphi$ in the derivation, and the induction hypothesis applies to the part of the derivation ending in those formulas (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \to \varphi$. Then $\Gamma \vDash \varphi$ by **??**.

Now suppose the last step is justified by QR. Then that step has the form $\chi \to \forall x\, B(x)$ and there is a preceding step $\chi \to \psi(c)$ with c not in Γ, χ, or $\forall x\, B(x)$. By induction hypothesis, $\Gamma \vDash \chi \to \forall x\, B(x)$. By **??**, $\Gamma \cup \{\chi\} \vDash \psi(c)$.

Consider some structure $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \models \forall x\, \psi(x)$. Since $\forall x\, \psi(x)$ is a sentence, this means we have to show that for every variable assignment s, $\mathfrak{M}, s \models \psi(x)$ (**??**). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \models \theta$ for all $\theta \in \Gamma$ by **??**. Let $\mathfrak{M}'$ be like $\mathfrak{M}$ except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ, $\mathfrak{M}' \models \Gamma \cup \{\chi\}$ by **??**. Since $\Gamma \cup \{\chi\} \vDash \psi(c)$, $\mathfrak{M}' \models B(c)$. Since $\psi(c)$ is a sentence, $\mathfrak{M}, s \models \psi(c)$ by **??**. $\mathfrak{M}', s \models \psi(x)$ iff $\mathfrak{M}' \models \psi(c)$ by **??** (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So, $\mathfrak{M}', s \models \psi(x)$. Since c does not occur in $\psi(x)$, by **??**, $\mathfrak{M}, s \models \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \models \forall x\, \psi(x)$. Thus $\Gamma \cup \{\chi\} \vDash \forall x\, \psi(x)$. By **??**, $\Gamma \vDash \chi \to \forall x\, \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x\, \psi(x) \to \chi$ is left as an exercise. □

Problem axd.7. Complete the proof of Theorem axd.36.

fol:axd:sou: cor:weak-soundness

Corollary axd.37. *If $\vdash \varphi$, then φ is valid.*

fol:axd:sou: cor:consistency-soundness

Corollary axd.38. *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from Γ. By Theorem axd.36, any structure $\mathfrak{M}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{M} \not\models \bot$ for every structure $\mathfrak{M}$, no $\mathfrak{M}$ can satisfy Γ, i.e., Γ is not satisfiable. □

axd.13 Derivations with Identity predicate

In order to accommodate $=$ in derivations, we simply add new axiom schemas. The definition of derivation and $\vdash$ remains the same, we just also allow the new axioms. fol:axd:ide:sec

Definition axd.39 (Axioms for identity predicate)**.**

$$t = t, \tag{axd.17}$$ fol:axd:ide:ax:id1

$$t_1 = t_2 \to (\psi(t_1) \to \psi(t_2)), \tag{axd.18}$$ fol:axd:ide:ax:id2

for any ground terms t, t_1, t_2.

Proposition axd.40. *The axioms eq. (axd.17) and eq. (axd.18) are valid.* *fol:axd:ide:prop:sound*

Proof. Exercise. □

Problem axd.8. Prove Proposition axd.40.

Proposition axd.41. *$\Gamma \vdash t = t$, for any term t and set Γ.* *fol:axd:ide:prop:iden1*

Proposition axd.42. *If $\Gamma \vdash \varphi(t_1)$ and $\Gamma \vdash t_1 = t_2$, then $\Gamma \vdash \varphi(t_2)$.* *fol:axd:ide:prop:iden2*

Proof. The formula

$$(t_1 = t_2 \to (\varphi(t_1) \to \varphi(t_2)))$$

is an instance of eq. (axd.18). The conclusion follows by two applications of MP. □

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Bibliography