

Chapter udf

Axiomatic Derivations

No effort has been made yet to ensure that the material in this chapter respects various tags indicating which connectives and quantifiers are primitive or defined: all are assumed to be primitive. If the FOL tag is true, we produce a version with quantifiers, otherwise without.

axd.1 Rules and Derivations

fol:axd:rul:
sec Axiomatic **derivations** are perhaps the simplest proof system for logic. A **derivation** explanation is just a sequence of **formulas**. To count as a **derivation**, every **formula** in the sequence must either be an instance of an axiom, or must follow from one or more **formulas** that precede it in the sequence by a rule of inference. A **derivation derives** its last **formula**.

Definition axd.1 (Derivability). If Γ is a set of **formulas** of \mathcal{L} then a **derivation** from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of **formulas** where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct **derivation** depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in is a correct inference step.

Definition axd.2 (Rule of inference). A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a **derivation** from Γ .

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a **derivation**, the following might be a very simple rule of inference:

If $\varphi \in \Gamma$, then φ is always a correct inference step in any **derivation** from Γ .

Similarly, if φ is one of the axioms, then φ by itself is a **derivation**, and so this is also a rule of inference:

If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to **formulas** that appear before the step considered. The following rule is called *modus ponens*:

If $\psi \rightarrow \varphi$ and ψ occur higher up in the **derivation**, then φ is a correct inference step.

If this is the only rule of inference, then our definition of **derivation** above amounts to this: $\varphi_1, \dots, \varphi_n$ is a **derivation** iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. for some $j < i$, φ_j is $\psi \rightarrow \varphi_i$, and for some $k < i$, φ_k is ψ .

The last clause says that φ_i follows from φ_j (ψ) and φ_k ($\psi \rightarrow \varphi_i$) by modus ponens. If we can go from 1 to n , and each time we find a **formula** φ_i that is either in Γ , an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct **derivation**.

Definition axd.3 (Derivability). A **formula** φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a **derivation** from Γ ending in φ .

Definition axd.4 (Theorems). A **formula** φ is a *theorem* if there is a **derivation** of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

axd.2 Axiom and Rules for the Propositional Connectives

Definition axd.5 (Axioms). The set of Ax_0 of *axioms* for the propositional connectives comprises all **formulas** of the following forms:

fol:axd:prp:	$(\varphi \wedge \psi) \rightarrow \varphi$	(axd.1)
ax:land1 fol:axd:prp:	$(\varphi \wedge \psi) \rightarrow \psi$	(axd.2)
ax:land2 fol:axd:prp:	$\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$	(axd.3)
ax:land3 fol:axd:prp:	$\varphi \rightarrow (\varphi \vee \psi)$	(axd.4)
ax:lor1 fol:axd:prp:	$\varphi \rightarrow (\psi \vee \varphi)$	(axd.5)
ax:lor2 fol:axd:prp:	$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$	(axd.6)
ax:lor3 fol:axd:prp:	$\varphi \rightarrow (\psi \rightarrow \varphi)$	(axd.7)
ax:lif1 fol:axd:prp:	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	(axd.8)
ax:lif2 fol:axd:prp:	$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$	(axd.9)
ax:lnot1 fol:axd:prp:	$\neg\varphi \rightarrow (\varphi \rightarrow \psi)$	(axd.10)
ax:lnot2 fol:axd:prp:	\top	(axd.11)
ax:ltrue fol:axd:prp:	$\perp \rightarrow \varphi$	(axd.12)
ax:lfalse1 fol:axd:prp:	$(\varphi \rightarrow \perp) \rightarrow \neg\varphi$	(axd.13)
ax:lfalse2 fol:axd:prp:	$\neg\neg\varphi \rightarrow \varphi$	(axd.14)
ax:dne		

Definition axd.6 (Modus ponens). If ψ and $\psi \rightarrow \varphi$ already occur in a derivation, then φ is a correct inference step.

We'll abbreviate the rule modus ponens as “MP.”

axd.3 Axioms and Rules for Quantifiers

fol:axd:qua:
sec

Definition axd.7 (Axioms for quantifiers). The *axioms* governing quantifiers are all instances of the following:

fol:axd:qua:	$\forall x \psi \rightarrow \psi(t),$	(axd.15)
ax:q1 fol:axd:qua:	$\psi(t) \rightarrow \exists x \psi.$	(axd.16)
ax:q2		

for any ground term t .

Definition axd.8 (Rules for quantifiers).

If $\psi \rightarrow \varphi(a)$ already occurs in the **derivation** and a does not occur in Γ or ψ , then $\psi \rightarrow \forall x \varphi(x)$ is a correct inference step.

If $\varphi(a) \rightarrow \psi$ already occurs in the **derivation** and a does not occur in Γ or ψ , then $\exists x \varphi(x) \rightarrow \psi$ is a correct inference step.

We'll abbreviate either of these by “QR.”

axd.4 Examples of Derivations

Example axd.9. Suppose we want to prove $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to **derive** it. Our only rule is MP, which given φ and $\varphi \rightarrow \psi$ allows us to justify ψ . One strategy would be to use [eq. \(axd.6\)](#) with φ being $\neg\theta$, ψ being α , and χ being $\theta \rightarrow \alpha$, i.e., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ is an instance of [eq. \(axd.10\)](#), and $\alpha \rightarrow (\theta \rightarrow \alpha)$ is an instance of [eq. \(axd.7\)](#). So our derivation is:

- | | |
|--|-------------|
| 1. $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ | eq. (axd.7) |
| 2. $(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$
$((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ | eq. (axd.6) |
| 3. $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ | 1, 2, MP |
| 4. $\alpha \rightarrow (\theta \rightarrow \alpha)$ | eq. (axd.7) |
| 5. $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$ | 3, 4, MP |

Example axd.10. Let's try to find a **derivation** of $\theta \rightarrow \theta$. It is not an instance of an axiom, so we have to use MP to **derive** it. [eq. \(axd.7\)](#) is an axiom of the form $\varphi \rightarrow \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \rightarrow \theta$, since this is what we want to **derive**. That means φ would also have to be θ , i.e., we might look at this instance of [eq. \(axd.7\)](#):

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ . But in our case, that would be θ , and we won't be able to **derive** θ by itself. So we need a different strategy.

The other axiom involving just \rightarrow is [eq. \(axd.8\)](#), i.e.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of [eq. \(axd.8\)](#) where $\varphi \rightarrow \chi$ is $\theta \rightarrow \theta$, the **formula** we are aiming for. Then of course, φ and χ are both θ . How should we pick ψ so that both $\varphi \rightarrow (\psi \rightarrow \chi)$ and $\varphi \rightarrow \psi$, i.e., in our case $\theta \rightarrow (\psi \rightarrow \theta)$ and $\theta \rightarrow \psi$, are also **derivable**? Well, the first of these is already an instance of [eq. \(axd.7\)](#), whatever we decide ψ to be. And $\theta \rightarrow \psi$ would be another instance of [eq. \(axd.7\)](#) if ψ were $(\theta \rightarrow \theta)$. So, our derivation is:

1. $\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$ eq. (axd.7)
2. $(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$
 $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$ eq. (axd.8)
3. $(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$ 1, 2, MP
4. $\theta \rightarrow (\theta \rightarrow \theta)$ eq. (axd.7)
5. $\theta \rightarrow \theta$ 3, 4, MP

Example axd.11. Sometimes we want to show that there is a derivation of some formula from some other formulas Γ . For instance, let's show that we can derive $\varphi \rightarrow \chi$ from $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$.

fol:axd:pro:
ex:chain

1. $\varphi \rightarrow \psi$ HYP
2. $\psi \rightarrow \chi$ HYP
3. $(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ eq. (axd.7)
4. $\varphi \rightarrow (\psi \rightarrow \chi)$ 2, 3, MP
5. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$
 $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ eq. (axd.8)
6. $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ 4, 5, MP
7. $\varphi \rightarrow \chi$ 1, 6, MP

The lines labelled “HYP” (for “hypothesis”) indicate that the formula on that line is an element of Γ .

Proposition axd.12. If $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$, then $\Gamma \vdash \varphi \rightarrow \chi$

fol:axd:pro:
prop:chain

Proof. Suppose $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$. Then there is a derivation of $\varphi \rightarrow \psi$ from Γ ; and a derivation of $\psi \rightarrow \chi$ from Γ as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of $\varphi \rightarrow \chi$ —which is the last line of the new derivation—from Γ . Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of $\varphi \rightarrow \psi$, and reference to line number 1 by reference to the last line of the derivation of $B \rightarrow \chi$. \square

Problem axd.1. Show that the following hold by exhibiting derivations from the axioms:

1. $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3. $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

axd.5 Derivations with Quantifiers

fol:axd:prq:
sec

Example axd.13. Let us give a derivation of $(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x (\varphi(x) \wedge \psi(x))$.

First, note that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x \varphi(x)$$

is an instance of eq. (axd.1), and

$$\forall x \varphi(x) \rightarrow \varphi(a)$$

of eq. (axd.15). So, by Proposition axd.12, we know that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \varphi(a)$$

is *derivable*. Likewise, since

$$\begin{aligned} (\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall y \psi(y) \quad \text{and} \\ \forall y \psi(y) \rightarrow \psi(a) \end{aligned}$$

are instances of eq. (axd.2) and eq. (axd.15), respectively,

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \psi(a)$$

is derivable by Proposition axd.12. Using an appropriate instance of eq. (axd.3) and two applications of MP, we see that

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow (\varphi(a) \wedge \psi(a))$$

is derivable. We can now apply QR to obtain

$$(\forall x \varphi(x) \wedge \forall y \psi(y)) \rightarrow \forall x (\varphi(x) \wedge \psi(x)).$$

axd.6 Proof-Theoretic Notions

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of *sentences* in *structures*, but by appeal to the *derivability* or *non-derivability* of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. fol:axd:ptn:sec

Definition axd.14 (Derivability). A formula φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a *derivation* from Γ ending in φ .

Definition axd.15 (Theorems). A formula φ is a *theorem* if there is a *derivation* of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition axd.16 (Consistency). A set Γ of **formulas** is *consistent* if and only if $\Gamma \not\vdash \perp$; it is *inconsistent* otherwise.

*fol:axd:ptn:
prop:reflexivity*

Proposition axd.17 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The **formula** φ by itself is a **derivation** of φ from Γ . □

*fol:axd:ptn:
prop:monotony*

Proposition axd.18 (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any **derivation** of φ from Γ is also a **derivation** of φ from Δ . □

*fol:axd:ptn:
prop:transitivity*

Proposition axd.19 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a **derivation** $\psi_1, \dots, \psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivation will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a **derivation** of φ from Γ , i.e., a **derivation** $\varphi_1, \dots, \varphi_k = \varphi$ where every φ_i is an axiom, an **element** of Γ , or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct **derivation** of ψ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

*fol:axd:ptn:
prop:incons*

Proposition axd.20. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every φ .*

Proof. Exercise. □

Problem axd.2. Prove [Proposition axd.20](#).

*fol:axd:ptn:
prop:proves-compact*

Proposition axd.21 (Compactness).

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite sequence of **formulas** $\varphi_1, \dots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, an **element** of Γ or follows from previous **formulas** by modus ponens. Take Γ_0 to be those φ_i which are in Γ . Then the **derivation** is likewise a **derivation** from Γ_0 , and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. □

axd.7 The Deduction Theorem

As we've seen, giving **derivations** in an axiomatic system is cumbersome, and **derivations** may be hard to find. Rather than actually write out long lists of **formulas**, it is generally easier to argue that such **derivations** exist, by making use of a few simple results. We've already established three such results: [Proposition axd.17](#) says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. [Proposition axd.18](#) says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And [Proposition axd.19](#) implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens:

Proposition axd.22. *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$, then $\Gamma \vdash \psi$.*

fol:axd:ded:
sec

fol:axd:ded:
prop:mp

Proof. We have that $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$:

1. φ Hyp.
2. $\varphi \rightarrow \psi$ Hyp.
3. ψ 1, 2, MP

By [Proposition axd.19](#), $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

Theorem axd.23 (Deduction Theorem). *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \rightarrow \psi$.*

fol:axd:ded:
thm:deduction-thm

Proof. The "if" direction is immediate. If $\Gamma \vdash \varphi \rightarrow \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \rightarrow \psi$ by [Proposition axd.18](#). Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by [Proposition axd.17](#). So, by [Proposition axd.22](#), $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the **derivation** of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every **derivation** of length 1. A **derivation** of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \rightarrow (\varphi \rightarrow \psi)$ by [eq. \(axd.7\)](#), and [Proposition axd.22](#) gives $\Gamma \vdash \varphi \rightarrow \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \rightarrow \psi$ because then last **sentence** $\varphi \rightarrow \psi$ is the same as $\varphi \rightarrow \varphi$, and we have **derived** that in [Example axd.10](#).

For the inductive step, suppose a **derivation** of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the **derivation** are $\chi \rightarrow \psi$ and χ , for some **formula** χ , i.e., $\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\begin{aligned} \Gamma \vdash \varphi \rightarrow (\chi \rightarrow \psi); \\ \Gamma \vdash \varphi \rightarrow \chi. \end{aligned}$$

But also

$$\Gamma \vdash (\varphi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)),$$

by eq. (axd.8), and two applications of Proposition axd.22 give $\Gamma \vdash \varphi \rightarrow \psi$, as required. \square

Notice how eq. (axd.7) and eq. (axd.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about [derivability](#), which we leave as exercises.

Proposition axd.24.

*fol:axd:ded:
prop:derivfacts*

*fol:axd:ded:
derivfacts:a*

1. $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi));$

*fol:axd:ded:
derivfacts:b*

2. *If $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ then $\Gamma \cup \{\psi\} \vdash \varphi$ (Contraposition);*

*fol:axd:ded:
derivfacts:c*

3. $\{\varphi, \neg\varphi\} \vdash \psi$ (*Ex Falso Quodlibet, Explosion*);

*fol:axd:ded:
derivfacts:d*

4. $\{\neg\neg\varphi\} \vdash \varphi$ (*Double Negation Elimination*);

*fol:axd:ded:
derivfacts:e*

5. *If $\Gamma \vdash \neg\neg\varphi$ then $\Gamma \vdash \varphi$;*

Problem axd.3. Prove Proposition axd.24

axd.8 The Deduction Theorem with Quantifiers

fol:axd:ddq:

*sec
fol:axd:ddq:*

thm:deduction-thm-q

Theorem axd.25 (Deduction Theorem). *If $\Gamma \cup \{\varphi\} \vdash \psi$ and then $\Gamma \vdash \varphi \rightarrow \psi$.*

Proof. We again proceed by induction on the length of the [derivation](#) of ψ from $\Gamma \cup \{\varphi\}$.

The proof of the induction basis is identical to that in the proof of [Theorem axd.23](#).

For the inductive step, suppose again that the [derivation](#) of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of [Theorem axd.23](#). If the inference rule is QR, we know that $\psi \equiv \chi \rightarrow \forall x \theta(x)$ and a [formula](#) of the form $\chi \rightarrow \theta(a)$ appears earlier in the [derivation](#), where a does not occur in χ , φ , or Γ . We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \theta(a)$$

and the induction hypothesis applies, i.e., we have that

$$\Gamma \vdash \varphi \rightarrow \theta(a)$$

By

$$\vdash (\varphi \rightarrow (\chi \rightarrow \theta(a))) \rightarrow ((\varphi \wedge \chi) \rightarrow \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \theta(a).$$

Since the eigenvariable condition still applies, we can add a step to this [derivation](#) justified by QR, and get:

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \forall x \theta(x)$$

We also have

$$\vdash ((\varphi \wedge \chi) \rightarrow \forall x \theta(x)) \rightarrow (\varphi \rightarrow (\chi \rightarrow \forall x \theta(x)))$$

so by modus ponens,

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \forall x \theta(x))$$

i.e., $\Gamma \vdash \psi$.

We leave the case where ψ is justified by the rule QR, but is of the form $\exists x \theta(x) \rightarrow \chi$, as an exercise. \square

Problem axd.4. Complete the proof of [Theorem axd.25](#).

axd.9 Derivability and Consistency

We will now establish a number of properties of the [derivability](#) relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. [fol:axd:prv:sec](#)

Proposition axd.26. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.* [fol:axd:prv:prop:provability-contr](#)

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \perp$. By [Proposition axd.17](#), $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By [Proposition axd.19](#), $\Gamma \vdash \perp$, i.e., Γ is inconsistent. \square

Proposition axd.27. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* [fol:axd:prv:prop:prov-incons](#)

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by [Proposition axd.18](#). $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by [Proposition axd.17](#). We also have $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by [eq. \(axd.10\)](#). So by two applications of [Proposition axd.22](#), we have $\Gamma \cup \{\neg\varphi\} \vdash \perp$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \perp$. By the deduction theorem, $\Gamma \vdash \neg\varphi \rightarrow \perp$. $\Gamma \vdash (\neg\varphi \rightarrow \perp) \rightarrow \neg\neg\varphi$ by [eq. \(axd.13\)](#), so $\Gamma \vdash \neg\neg\varphi$ by [Proposition axd.22](#). Since $\Gamma \vdash \neg\neg\varphi \rightarrow \varphi$ ([eq. \(axd.14\)](#)), we have $\Gamma \vdash \varphi$ by [Proposition axd.22](#) again. \square

Problem axd.5. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:axd:prv: **Proposition axd.28.** *prop:explicit-inc* If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.

Proof. $\Gamma \vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by eq. (axd.10). $\Gamma \vdash \perp$ by two applications of Proposition axd.22. \square

fol:axd:prv: **Proposition axd.29.** *prop:provability-exhaustive* If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.

Proof. Exercise. \square

Problem axd.6. Prove Proposition axd.29

axd.10 Derivability and the Propositional Connectives

fol:axd:ppr:
fol:axd:ppr:^{sec}
prop:provability-land

Proposition axd.30.

fol:axd:ppr:
prop:provability-land-left
fol:axd:ppr:
prop:provability-land-right

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$
2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. From eq. (axd.1) and eq. (axd.1) by modus ponens.
2. From eq. (axd.3) by two applications of modus ponens. \square

fol:axd:ppr:
prop:provability-lor

Proposition axd.31.

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.
2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. From eq. (axd.9) we get $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ and $\vdash \neg\psi \rightarrow (\psi \rightarrow \perp)$. So by the deduction theorem, we have $\{\neg\varphi\} \vdash \varphi \rightarrow \perp$ and $\{\neg\psi\} \vdash \psi \rightarrow \perp$. From eq. (axd.6) we get $\{\neg\varphi, \neg\psi\} \vdash (\varphi \vee \psi) \rightarrow \perp$. By the deduction theorem, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \vdash \perp$.

2. From eq. (axd.4) and eq. (axd.5) by modus ponens. \square

fol:axd:ppr:
prop:provability-lif

Proposition axd.32.

fol:axd:ppr:
prop:provability-lif-left
fol:axd:ppr:
prop:provability-lif-right

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.
2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

Proof. 1. We can derive:

1. φ HYP
2. $\varphi \rightarrow \psi$ HYP
3. ψ 1, 2, MP

2. By eq. (axd.10) and eq. (axd.7) and the deduction theorem, respectively. \square

axd.11 Derivability and the Quantifiers

Theorem axd.33. *If c is a constant symbol not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x \varphi(x)$.*

fol:axd:qpr:
sec
fol:axd:qpr:
thm:strong-generalization

Proof. By the deduction theorem, $\Gamma \vdash \top \rightarrow \varphi(c)$. Since c does not occur in Γ or \top , we get $\Gamma \vdash \top \rightarrow \varphi(c)$. By the deduction theorem again, $\Gamma \vdash \forall x \varphi(x)$. \square

Proposition axd.34.

fol:axd:qpr:
prop:provability-quantifiers

1. $\varphi(t) \vdash \exists x \varphi(x)$.
2. $\forall x \varphi(x) \vdash \varphi(t)$.

Proof. 1. By eq. (axd.16) and the deduction theorem.

2. By eq. (axd.15) and the deduction theorem.

\square

axd.12 Soundness

explanation A derivation system, such as axiomatic deduction, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

fol:axd:sou:
sec

1. every derivable φ is valid;
2. if φ is derivable from some others Γ , it is also a consequence of them;
3. if a set of formulas Γ is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Proposition axd.35. *If φ is an axiom, then $\mathfrak{M}, s \models \varphi$ for each structure \mathfrak{M} and assignment s .*

Proof. We have to verify that all the axioms are valid. For instance, here is the case for eq. (axd.15): suppose t is **free for** x in φ , and assume $\mathfrak{M}, s \models \forall x \varphi$. Then by definition of satisfaction, for each $s' \sim_x s$, also $\mathfrak{M}, s' \models \varphi$, and in particular this holds when $s'(x) = \text{Val}_s^{\mathfrak{M}}(t)$. By ??, $\mathfrak{M}, s \models \varphi[t/x]$. This shows that $\mathfrak{M}, s \models (\forall x \varphi \rightarrow \varphi[t/x])$. \square

fol:axd:sou:
thm:soundness

Theorem axd.36 (Soundness). *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Proof. By induction on the length of the **derivation** of φ from Γ . If there are no steps justified by inferences, then all **formulas** in the derivation are either instances of axioms or are in Γ . By the previous proposition, all the axioms are valid, and hence if φ is an axiom then $\Gamma \models \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \models \varphi$.

If the last step of the derivation of φ is justified by modus ponens, then there are **formulas** ψ and $\psi \rightarrow \varphi$ in the **derivation**, and the induction hypothesis applies to the part of the **derivation** ending in those **formulas** (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \models \psi$ and $\Gamma \models \psi \rightarrow \varphi$. Then $\Gamma \models \varphi$ by ??.

Now suppose the last step is justified by QR. Then that step has the form $\chi \rightarrow \forall x B(x)$ and there is a preceding step $\chi \rightarrow \psi(c)$ with c not in Γ , χ , or $\forall x B(x)$. By induction hypothesis, $\Gamma \models \chi \rightarrow \forall x B(x)$. By ??, $\Gamma \cup \{\chi\} \models \psi(c)$.

Consider some structure \mathfrak{M} such that $\mathfrak{M} \models \Gamma \cup \{\chi\}$. We need to show that $\mathfrak{M} \models \forall x \psi(x)$. Since $\forall x \psi(x)$ is a **sentence**, this means we have to show that for every variable assignment s , $\mathfrak{M}, s \models \psi(x)$ (??). Since $\Gamma \cup \{\chi\}$ consists entirely of sentences, $\mathfrak{M}, s \models \theta$ for all $\theta \in \Gamma$ by ??. Let \mathfrak{M}' be like \mathfrak{M} except that $c^{\mathfrak{M}'} = s(x)$. Since c does not occur in Γ or χ , $\mathfrak{M}' \models \Gamma \cup \{\chi\}$ by ??. Since $\Gamma \cup \{\chi\} \models \psi(c)$, $\mathfrak{M}' \models \psi(c)$. Since $\psi(c)$ is a **sentence**, $\mathfrak{M}, s \models \psi(c)$ by ??. $\mathfrak{M}', s \models \psi(x)$ iff $\mathfrak{M}' \models \psi(c)$ by ?? (recall that $\psi(c)$ is just $\psi(x)[c/x]$). So, $\mathfrak{M}', s \models \psi(x)$. Since c does not occur in $\psi(x)$, by ??, $\mathfrak{M}, s \models \psi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \models \forall x \psi(x)$. Thus $\Gamma \cup \{\chi\} \models \forall x \psi(x)$. By ??, $\Gamma \models \chi \rightarrow \forall x \psi(x)$.

The case where φ is justified by QR but is of the form $\exists x \psi(x) \rightarrow \chi$ is left as an exercise. \square

Problem axd.7. Complete the proof of [Theorem axd.36](#).

fol:axd:sou:
cor:weak-soundness

Corollary axd.37. *If $\vdash \varphi$, then φ is valid.*

fol:axd:sou:
cor:consistency-soundness

Corollary axd.38. *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \perp$, i.e., there is a **derivation** of \perp from Γ . By [Theorem axd.36](#), any **structure** \mathfrak{M} that satisfies Γ must satisfy \perp . Since $\mathfrak{M} \not\models \perp$ for every **structure** \mathfrak{M} , no \mathfrak{M} can satisfy Γ , i.e., Γ is not satisfiable. \square

axd.13 Derivations with Identity predicate

In order to accommodate $=$ in **derivations**, we simply add new axiom schemas. fol:axd:ide:sec
The definition of **derivation** and \vdash remains the same, we just also allow the new axioms.

Definition axd.39 (Axioms for **identity predicate**).

$$t = t, \quad (\text{axd.17}) \quad \text{fol:axd:ide:ax:id1}$$

$$t_1 = t_2 \rightarrow (\psi(t_1) \rightarrow \psi(t_2)), \quad (\text{axd.18}) \quad \text{fol:axd:ide:ax:id2}$$

for any ground terms t, t_1, t_2 .

Proposition axd.40. *The axioms eq. (axd.17) and eq. (axd.18) are valid.* fol:axd:ide:prop:sound

Proof. Exercise. □

Problem axd.8. Prove **Proposition axd.40**.

Proposition axd.41. $\Gamma \vdash t = t$, for any term t and set Γ .

fol:axd:ide:prop:iden1

Proposition axd.42. If $\Gamma \vdash \varphi(t_1)$ and $\Gamma \vdash t_1 = t_2$, then $\Gamma \vdash \varphi(t_2)$.

fol:axd:ide:prop:iden2

Proof. The **formula**

$$(t_1 = t_2 \rightarrow (\varphi(t_1) \rightarrow \varphi(t_2)))$$

is an instance of **eq. (axd.18)**. The conclusion follows by two applications of MP. □

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Bibliography