min.1 Sphere Models

One way of providing a formal semantics for counterfactuals is to turn Lewis’s informal account into a mathematical structure. The spheres around a world \( w \) then are sets of worlds. Since the spheres are nested, the sets of worlds around \( w \) have to be linearly ordered by the subset relation.

**Definition min.1.** A sphere model is a triple \( \mathfrak{M} = \langle W, O, V \rangle \) where \( W \) is a non-empty set of worlds, \( V : \text{At}_0 \to \wp(W) \) is a valuation, and \( O : W \to \wp(\wp(W)) \) assigns to each world \( w \) a system of spheres \( O_w \). For each \( w, O_w \) is a set of sets of worlds, and must satisfy:

1. \( O_w \) is centered on \( w \): \( \{ w \} \in O_w \).
2. \( O_w \) is nested: whenever \( S_1, S_2 \in O_w, S_1 \subseteq S_2 \) or \( S_2 \subseteq S_1 \), i.e., \( O_w \) is linearly ordered by \( \subseteq \).
3. \( O_w \) is closed under non-empty unions.
4. \( O_w \) is closed under non-empty intersections.

The intuition behind \( O_w \) is that the worlds “around” \( w \) are stratified according to how far away they are from \( w \). The innermost sphere is just \( w \) by itself, i.e., the set \( \{ w \} \): \( w \) is closer to \( w \) than the worlds in any other sphere. If \( S \subseteq S' \), then the worlds in \( S' \setminus S \) are further way from \( w \) than the worlds in \( S \): \( S' \setminus S \) is the “layer” between the \( S \) and the worlds outside of \( S' \). In particular, we have to think of the spheres as containing all the worlds within their outer surface; they are not just the individual layers.

The diagram in Figure 1 corresponds to the sphere model with \( W = \{ w, w_1, \ldots, w_7 \}, V(p) = \{ w_5, w_6, w_7 \} \). The innermost sphere \( S_1 = \{ w \} \). The closest worlds to \( w \) are \( w_1, w_2, w_3 \), so the next larger sphere is \( S_2 = \{ w, w_1, w_2, w_3 \} \).

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**Figure 1: Diagram of a sphere model**

![Diagram of a sphere model](sphere-models)
The worlds further out are $w_4$, $w_5$, $w_6$, so the outermost sphere is $S_3 = \{w, w_1, \ldots, w_6\}$. The system of spheres around $w$ is $O_w = \{S_1, S_2, S_3\}$. The world $w_7$ is not in any sphere around $w$. The closest worlds in which $p$ is true are $w_5$ and $w_6$, and so the smallest $p$-admitting sphere is $S_3$.

To define satisfaction of a formula $\varphi$ at world $w$ in a sphere model $\mathcal{M}$, $\mathcal{M}, w \models \varphi$, we expand the definition for modal formulas to include a clause for $\psi \rightarrow \chi$:

**Definition min.2.** $\mathcal{M}, w \models \psi \rightarrow \chi$ iff either

1. For all $u \in \bigcup O_w$, $\mathcal{M}, u \not\models \chi$, or
2. For some $S \in O_w$,
   a) $\mathcal{M}, u \models \psi$ for some $u \in S$, and
   b) for all $v \in S$, either $\mathcal{M}, v \not\models \psi$ or $\mathcal{M}, v \models \chi$.

According to this definition, $\mathcal{M}, w \models \psi \rightarrow \chi$ iff either the antecedent $\psi$ is false everywhere in the spheres around $w$, or there is a sphere $S$ where $\psi$ is true, and the material conditional $\psi \rightarrow \chi$ is true at all worlds in that “$\psi$-admitting” sphere. Note that we didn’t require in the definition that $S$ is the *innermost* $\psi$-admitting sphere, contrary to what one might expect from the intuitive explanation. But if the condition in (2) is satisfied for some sphere $S$, then it is also satisfied for all spheres $S$ contains, and hence in particular for the innermost sphere.

Note also that the definition of sphere models does not require that there is an innermost $\psi$-admitting sphere: we may have an infinite sequence $S_1 \supseteq S_2 \supseteq \cdots \supseteq \{w\}$ of $\psi$-admitting spheres, and hence no innermost $\psi$-admitting spheres. In that case, $\mathcal{M}, w \models \psi \rightarrow \chi$ iff $\psi \rightarrow \chi$ holds throughout the spheres $S_i$, $S_{i+1}$, \ldots, for some $i$.

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**Bibliography**