

int.1 The Material Conditional

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sec In its simplest form in English, a conditional is a sentence of the form “If ... then ...,” where the ... are themselves sentences, such as “If the butler did it, then the gardener is innocent.” In introductory logic courses, we learn to symbolize conditionals using the \rightarrow connective: symbolize the parts indicated by ..., e.g., by **formulas** φ and ψ , and the entire conditional is symbolized by $\varphi \rightarrow \psi$.

The connective \rightarrow is *truth-functional*, i.e., the truth value— \mathbb{T} or \mathbb{F} —of $\varphi \rightarrow \psi$ is determined by the truth values of φ and ψ : $\varphi \rightarrow \psi$ is true iff φ is false or ψ is true, and false otherwise. Relative to a truth value assignment \mathfrak{v} , we define $\mathfrak{v} \models \varphi \rightarrow \psi$ iff $\mathfrak{v} \not\models \varphi$ or $\mathfrak{v} \models \psi$. The connective \rightarrow with this semantics is called the *material conditional*.

This definition results in a number of elementary logical facts. First of all, the deduction theorem holds for the material conditional:

$$\text{If } \Gamma, \varphi \models \psi \text{ then } \Gamma \models \varphi \rightarrow \psi \quad (1)$$

It is truth-functional: $\varphi \rightarrow \psi$ and $\neg\varphi \vee \psi$ are equivalent:

$$\varphi \rightarrow \psi \models \neg\varphi \vee \psi \quad (2)$$

$$\neg\varphi \vee \psi \models \varphi \rightarrow \psi \quad (3)$$

A material conditional is entailed by its consequent and by the negation of its antecedent:

$$\psi \models \varphi \rightarrow \psi \quad (4)$$

$$\neg\varphi \models \varphi \rightarrow \psi \quad (5)$$

A false material conditional is equivalent to the conjunction of its antecedent and the negation of its consequent: if $\varphi \rightarrow \psi$ is false, $\varphi \wedge \neg\psi$ is true, and vice versa:

$$\neg(\varphi \rightarrow \psi) \models \varphi \wedge \neg\psi \quad (6)$$

$$\varphi \wedge \neg\psi \models \neg(\varphi \rightarrow \psi) \quad (7)$$

The material conditional supports modus ponens:

$$\varphi, \varphi \rightarrow \psi \models \psi \quad (8)$$

The material conditional agglomerates:

$$\varphi \rightarrow \psi, \varphi \rightarrow \chi \models \varphi \rightarrow (\psi \wedge \chi) \quad (9)$$

We can always strengthen the antecedent, i.e., the conditional is *monotonic*:

$$\varphi \rightarrow \psi \models (\varphi \wedge \chi) \rightarrow \psi \quad (10)$$

The material conditional is transitive, i.e., the chain rule is valid:

$$\varphi \rightarrow \psi, \psi \rightarrow \chi \vDash \varphi \rightarrow \chi \quad (11)$$

The material conditional is equivalent to its contrapositive:

$$\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \neg\varphi \quad (12)$$

$$\neg\psi \rightarrow \neg\varphi \vDash \varphi \rightarrow \psi \quad (13)$$

These are all useful and unproblematic inferences in mathematical reasoning. However, the philosophical and linguistic literature is replete with purported counterexamples to the equivalent inferences in non-mathematical contexts. These suggest that the material conditional \rightarrow is not—or at least not always—the appropriate connective to use when symbolizing English “if ... then ...” statements.

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Bibliography