int.1 The Material Conditional

In its simplest form in English, a conditional is a sentence of the form “If . . . then . . . ,” where the . . . are themselves sentences, such as “If the butler did it, then the gardener is innocent.” In introductory logic courses, we earn to symbolize conditionals using the $\rightarrow$ connective: symbolize the parts indicated by . . . , e.g., by formulas $\varphi$ and $\psi$, and the entire conditional is symbolized by $\varphi \rightarrow \psi$.

The connective $\rightarrow$ is truth-functional, i.e., the truth value $T$ or $F$ of $\varphi \rightarrow \psi$ is determined by the truth values of $\varphi$ and $\psi$: $\varphi \rightarrow \psi$ is true iff $\varphi$ is false or $\psi$ is true, and false otherwise. Relative to a truth value assignment $v$, we define $v \vDash \varphi \rightarrow \psi$ iff $v \not\vDash \varphi$ or $v \vDash \psi$. The connective $\rightarrow$ with this semantics is called the material conditional.

This definition results in a number of elementary logical facts. First of all, the deduction theorem holds for the material conditional:

If $\Gamma, \varphi \vDash \psi$ then $\Gamma \vDash \varphi \rightarrow \psi$ (1)

It is truth-functional: $\varphi \rightarrow \psi$ and $\neg \varphi \lor \psi$ are equivalent:

$\varphi \rightarrow \psi \vDash \neg \varphi \lor \psi$ (2)

$\neg \varphi \lor \psi \vDash \varphi \rightarrow \psi$ (3)

A material conditional is entailed by its consequent and by the negation of its antecedent:

$\psi \vDash \varphi \rightarrow \psi$ (4)

$\neg \varphi \vDash \varphi \rightarrow \psi$ (5)

A false material conditional is equivalent to the conjunction of its antecedent and the negation of its consequent: if $\varphi \rightarrow \psi$ is false, $\varphi \land \neg \psi$ is true, and vice versa:

$\neg (\varphi \rightarrow \psi) \vDash \varphi \land \neg \psi$ (6)

$\varphi \land \neg \psi \vDash \neg (\varphi \rightarrow \psi)$ (7)

The material conditional supports modus ponens:

$\varphi, \varphi \rightarrow \psi \vDash \psi$ (8)

The material conditional agglomerates:

$\varphi \rightarrow \psi, \varphi \rightarrow \chi \vDash \varphi \rightarrow (\psi \land \chi)$ (9)

We can always strengthen the antecedent, i.e., the conditional is monotonic:

$\varphi \rightarrow \psi \vDash (\varphi \land \chi) \rightarrow \psi$ (10)
The material conditional is transitive, i.e., the chain rule is valid:

\[ \varphi \rightarrow \psi, \psi \rightarrow \chi \models \varphi \rightarrow \chi \quad (11) \]

The material conditional is equivalent to its contrapositive:

\[ \varphi \rightarrow \psi \models \neg \psi \rightarrow \neg \varphi \quad (12) \]
\[ \neg \psi \rightarrow \neg \varphi \models \varphi \rightarrow \psi \quad (13) \]

These are all useful and unproblematic inferences in mathematical reasoning. However, the philosophical and linguistic literature is replete with purported counterexamples to the equivalent inferences in non-mathematical contexts. These suggest that the material conditional \( \rightarrow \) is not—or at least not always—the appropriate connective to use when symbolizing English “if . . . then . . .” statements.

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Bibliography