

# Chapter udf

## Introduction

### int.1 The Material Conditional

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sec In its simplest form in English, a conditional is a sentence of the form “If ... then ...,” where the ... are themselves sentences, such as “If the butler did it, then the gardener is innocent.” In introductory logic courses, we learn to symbolize conditionals using the  $\rightarrow$  connective: symbolize the parts indicated by ..., e.g., by **formulas**  $\varphi$  and  $\psi$ , and the entire conditional is symbolized by  $\varphi \rightarrow \psi$ .

The connective  $\rightarrow$  is *truth-functional*, i.e., the truth value— $\mathbb{T}$  or  $\mathbb{F}$ —of  $\varphi \rightarrow \psi$  is determined by the truth values of  $\varphi$  and  $\psi$ :  $\varphi \rightarrow \psi$  is true iff  $\varphi$  is false or  $\psi$  is true, and false otherwise. Relative to a truth value assignment  $\mathfrak{v}$ , we define  $\mathfrak{v} \models \varphi \rightarrow \psi$  iff  $\mathfrak{v} \not\models \varphi$  or  $\mathfrak{v} \models \psi$ . The connective  $\rightarrow$  with this semantics is called the *material conditional*.

This definition results in a number of elementary logical facts. First of all, the deduction theorem holds for the material conditional:

$$\text{If } \Gamma, \varphi \models \psi \text{ then } \Gamma \models \varphi \rightarrow \psi \quad (\text{int.1})$$

It is truth-functional:  $\varphi \rightarrow \psi$  and  $\neg\varphi \vee \psi$  are equivalent:

$$\varphi \rightarrow \psi \models \neg\varphi \vee \psi \quad (\text{int.2})$$

$$\neg\varphi \vee \psi \models \varphi \rightarrow \psi \quad (\text{int.3})$$

A material conditional is entailed by its consequent and by the negation of its antecedent:

$$\psi \models \varphi \rightarrow \psi \quad (\text{int.4})$$

$$\neg\varphi \models \varphi \rightarrow \psi \quad (\text{int.5})$$

A false material conditional is equivalent to the conjunction of its antecedent and the negation of its consequent: if  $\varphi \rightarrow \psi$  is false,  $\varphi \wedge \neg\psi$  is true, and vice versa:

$$\neg(\varphi \rightarrow \psi) \vDash \varphi \wedge \neg\psi \quad (\text{int.6})$$

$$\varphi \wedge \neg\psi \vDash \neg(\varphi \rightarrow \psi) \quad (\text{int.7})$$

The material conditional supports modus ponens:

$$\varphi, \varphi \rightarrow \psi \vDash \psi \quad (\text{int.8})$$

The material conditional agglomerates:

$$\varphi \rightarrow \psi, \varphi \rightarrow \chi \vDash \varphi \rightarrow (\psi \wedge \chi) \quad (\text{int.9})$$

We can always strengthen the antecedent, i.e., the conditional is *monotonic*:

$$\varphi \rightarrow \psi \vDash (\varphi \wedge \chi) \rightarrow \psi \quad (\text{int.10})$$

The material conditional is transitive, i.e., the chain rule is valid:

$$\varphi \rightarrow \psi, \psi \rightarrow \chi \vDash \varphi \rightarrow \chi \quad (\text{int.11})$$

The material conditional is equivalent to its contrapositive:

$$\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \neg\varphi \quad (\text{int.12})$$

$$\neg\psi \rightarrow \neg\varphi \vDash \varphi \rightarrow \psi \quad (\text{int.13})$$

These are all useful and unproblematic inferences in mathematical reasoning. However, the philosophical and linguistic literature is replete with purported counterexamples to the equivalent inferences in non-mathematical contexts. These suggest that the material conditional  $\rightarrow$  is not—or at least not always—the appropriate connective to use when symbolizing English “if ... then ...” statements.

## int.2 Paradoxes of the Material Conditional

One of the first to criticize the use of  $\varphi \rightarrow \psi$  as a way to symbolize “if ... then ...” statements of English was C. I. Lewis. Lewis was criticizing the use of the material condition in Whitehead and Russell’s *Principia Mathematica*, who pronounced  $\rightarrow$  as “implies.” Lewis rightly complained that if  $\rightarrow$  meant “implies,” then any false proposition  $p$  implies that  $p$  implies  $q$ , since  $p \rightarrow (p \rightarrow q)$  is true if  $p$  is false, and that any true proposition  $q$  implies that  $p$  implies  $q$ , since  $q \rightarrow (p \rightarrow q)$  is true if  $q$  is true.

Logicians of course know that *implication*, i.e., logical entailment, is not a connective but a relation between **formulas** or statements. So we should just

not read  $\rightarrow$  as “implies” to avoid confusion.<sup>1</sup> As long as we don’t, the particular worry that Lewis had simply does not arise:  $p$  does not “imply”  $q$  even if we think of  $p$  as standing for a false English sentence. To determine if  $p \models q$  we must consider *all valuations*, and  $p \not\models q$  even when we use  $p$  to symbolize a sentence which happens to be false.

But there is still something odd about “if ... then ...” statements such as Lewis’s

If the moon is made of green cheese, then  $2 + 2 = 4$ .

and about the inferences

The moon is not made of green cheese. Therefore, if the moon is made of green cheese, then  $2 + 2 = 4$ .

$2 + 2 = 4$ . Therefore, if the moon is made of green cheese, then  $2 + 2 = 4$ .

Yet, if “if ... then ...” were just  $\rightarrow$ , the sentence would be unproblematically true, and the inferences unproblematically valid.

Another example of concerns the tautology  $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ . This would suggest that if you take two indicative sentences  $S$  and  $T$  from the newspaper at random, the sentence “If  $S$  then  $T$ , or if  $T$  then  $S$ ” should be true.

### int.3 The Strict Conditional

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Lewis introduced the *strict conditional*  $\rightarrow$  and argued that it, not the material conditional, corresponds to implication. In alethic modal logic,  $\varphi \rightarrow \psi$  can be defined as  $\Box(\varphi \rightarrow \psi)$ . A strict conditional is thus true (at a world) iff the corresponding material conditional is necessary.

How does the strict conditional fare vis-a-vis the paradoxes of the material conditional? A strict conditional with a false antecedent and one with a true consequent, may be true, or it may be false. Moreover,  $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$  is not valid. The strict conditional  $\varphi \rightarrow \psi$  is also not equivalent to  $\neg\varphi \vee \psi$ , so it is not truth functional.

We have:

$$\varphi \rightarrow \psi \models \neg\varphi \vee \psi \text{ but:} \tag{int.14}$$

$$\neg\varphi \vee \psi \not\models \varphi \rightarrow \psi \tag{int.15}$$

$$\psi \not\models \varphi \rightarrow \psi \tag{int.16}$$

$$\neg\varphi \not\models \varphi \rightarrow \psi \tag{int.17}$$

$$\neg(\varphi \rightarrow \psi) \not\models \varphi \wedge \neg\psi \text{ but:} \tag{int.18}$$

$$\varphi \wedge \neg\psi \models \neg(\varphi \rightarrow \psi) \tag{int.19}$$

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<sup>1</sup>Reading “ $\rightarrow$ ” as “implies” is still widely practised by mathematicians and computer scientists, although philosophers try to avoid the confusions Lewis highlighted by pronouncing it as “only if.”

However, the strict conditional still supports modus ponens:

$$\varphi, \varphi \rightarrow \psi \vDash \psi \quad (\text{int.20})$$

The strict conditional agglomerates:

$$\varphi \rightarrow \psi, \varphi \rightarrow \chi \vDash \varphi \rightarrow (\psi \wedge \chi) \quad (\text{int.21})$$

Antecedent strengthening holds for the strict conditional:

$$\varphi \rightarrow \psi \vDash (\varphi \wedge \chi) \rightarrow \psi \quad (\text{int.22})$$

The strict conditional is also transitive:

$$\varphi \rightarrow \psi, \psi \rightarrow \chi \vDash \varphi \rightarrow \chi \quad (\text{int.23})$$

Finally, the strict conditional is equivalent to its contrapositive:

$$\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \neg\varphi \quad (\text{int.24})$$

$$\neg\psi \rightarrow \neg\varphi \vDash \varphi \rightarrow \psi \quad (\text{int.25})$$

**Problem int.1.** Give **S5**-counterexamples to the entailment relations which do not hold for the strict conditional, i.e., for:

1.  $\neg p \not\vDash \Box(p \rightarrow q)$
2.  $q \not\vDash \Box(p \rightarrow q)$
3.  $\neg\Box(p \rightarrow q) \not\vDash p \wedge \neg q$
4.  $\not\vDash \Box(p \rightarrow q) \vee \Box(q \rightarrow p)$

**Problem int.2.** Show that the valid entailment relations hold for the strict conditional by giving **S5**-proofs of:

1.  $\Box(\varphi \rightarrow \psi) \vDash \neg\varphi \vee \psi$
2.  $\varphi \wedge \neg\psi \vDash \neg\Box(\varphi \rightarrow \psi)$
3.  $\varphi, \Box(\varphi \rightarrow \psi) \vDash \psi$
4.  $\Box(\varphi \rightarrow \psi), \Box(\varphi \rightarrow \chi) \vDash \Box(\varphi \rightarrow (\psi \wedge \chi))$
5.  $\Box(\varphi \rightarrow \psi) \vDash \Box((\varphi \wedge \chi) \rightarrow \psi)$
6.  $\Box(\varphi \rightarrow \psi), \Box(\psi \rightarrow \chi) \vDash \Box(\varphi \rightarrow \chi)$
7.  $\Box(\varphi \rightarrow \psi) \vDash \Box(\neg\psi \rightarrow \neg\varphi)$
8.  $\Box(\neg\psi \rightarrow \neg\varphi) \vDash \Box(\varphi \rightarrow \psi)$

However, the strict conditional still has its own “paradoxes.” Just as a material conditional with a false antecedent or a true consequent is true, a strict conditional with a *necessarily* false antecedent or a necessarily true consequent is true. Moreover, any true strict conditional is necessarily true, and any false strict conditional is necessarily false. In other words, we have

$$\Box\neg\varphi \vDash \varphi \rightarrow \psi \quad (\text{int.26})$$

$$\Box\psi \vDash \varphi \rightarrow \psi \quad (\text{int.27})$$

$$\varphi \rightarrow \psi \vDash \Box(\varphi \rightarrow \psi) \quad (\text{int.28})$$

$$\neg(\varphi \rightarrow \psi) \vDash \Box\neg(\varphi \rightarrow \psi) \quad (\text{int.29})$$

These are not problems if you think of  $\rightarrow$  as “implies.” Logical entailment relationships are, after all, mathematical facts and so can’t be contingent. But they do raise issues if you want to use  $\rightarrow$  as a logical connective that is supposed to capture “if ... then ...,” especially the last two. For surely there are “if ... then ...” statements that are contingently true or contingently false—in fact, they generally are neither necessary nor impossible.

**Problem int.3.** Give proofs in **S5** of:

1.  $\Box\neg\psi \vDash \varphi \rightarrow \psi$
2.  $\varphi \rightarrow \psi \vDash \Box(\varphi \rightarrow \psi)$
3.  $\neg(\varphi \rightarrow \psi) \vDash \Box\neg(\varphi \rightarrow \psi)$

Use the definition of  $\rightarrow$  to do so.

## int.4 Counterfactuals

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sec A very common and important form of “if ... then ...” constructions in English are built using the past subjunctive form of *to be*: “if it were the case that ... then it would be the case that ...” Because usually the antecedent of such a conditional is false, i.e., counter to fact, they are called *counterfactual conditionals* (and because they use the subjunctive form of *to be*, also *subjunctive conditionals*). They are distinguished from *indicative conditionals* which take the form of “if it is the case that ... then it is the case that ...” Counterfactual and indicative conditionals differ in truth conditions. Consider Adams’s famous example:

If Oswald didn’t kill Kennedy, someone else did.

If Oswald hadn’t killed Kennedy, someone else would have.

The first is indicative, the second counterfactual. The first is clearly true: we know President John F. Kennedy was killed by *someone*, and if that someone wasn’t (contrary to the Warren Report) Lee Harvey Oswald, then someone else killed Kennedy. The second one says something different. It claims that

if Oswald hadn't killed Kennedy, i.e., if the Dallas shooting had been avoided or had been unsuccessful, history would have subsequently unfolded in such a way that another assassination would have been successful. In order for it to be true, it would have to be the case that powerful forces had conspired to ensure JFK's death (as many JFK conspiracy theorists believe).

It is a live debate whether the *indicative* conditional is correctly captured by the material conditional, in particular, whether the paradoxes of the material conditional can be "explained" in a way that is compatible with it giving the truth conditions for English indicative conditionals. By contrast, it is uncontroversial that counterfactual conditionals cannot be symbolized correctly by the material conditionals. That is clear because, even though generally the antecedents of counterfactuals are false, not all counterfactuals with false antecedents are true—for instance, if you believe the Warren Report, and there was no conspiracy to assassinate JFK, then Adams's counterfactual conditional is an example.

Counterfactual conditionals play an important role in causal reasoning: a prime example of the use of counterfactuals is to express causal relationships. E.g., striking a match causes it to light, and you can express this by saying "if this match were struck, it would light." Material, and generally indicative conditionals, cannot be used to express this: "the match is struck  $\rightarrow$  the match lights" is true if the match is never struck, regardless of what would happen if it were. Even worse, "the match is struck  $\rightarrow$  the match turns into a bouquet of flowers" is also true if it is never struck, but the match would certainly not turn into a bouquet of flowers if it were struck.

It is still debated What exactly the correct logic of counterfactuals is. An influential analysis of counterfactuals was given by Stalnaker and Lewis. According to them, a counterfactual "if it were the case that  $S$  then it would be the case that  $T$ " is true iff  $T$  is true in the counterfactual situation ("possible world") that is closest to the way the actual world is and where  $S$  is true. This is called an "ontic" analysis, since it makes reference to an ontology of possible worlds. Other analyses make use of conditional probabilities or theories of belief revision. There is a proliferation of different proposed logics of counterfactuals. There isn't even a single Lewis–Stalnaker logic of counterfactuals: even though Stalnaker and Lewis proposed accounts along similar lines with reference to closest possible worlds, the assumptions they made result in different valid inferences.

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# Bibliography