

## rec.1 Sequences

cmp:rec:seq:  
sec

The set of primitive recursive functions is remarkably robust. But we will be able to do even more once we have developed an adequate means of handling sequences. We will identify finite sequences of natural numbers with natural numbers in the following way: the sequence  $\langle a_0, a_1, a_2, \dots, a_k \rangle$  corresponds to the number

$$p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot \dots \cdot p_k^{a_k+1}.$$

We add one to the exponents to guarantee that, for example, the sequences  $\langle 2, 7, 3 \rangle$  and  $\langle 2, 7, 3, 0, 0 \rangle$  have distinct numeric codes. We can take both 0 and 1 to code the empty sequence; for concreteness, let  $\Lambda$  denote 0.

The reason that this coding of sequences works is the so-called Fundamental Theorem of Arithmetic: every natural number  $n \geq 2$  can be written in one and only one way in the form

$$n = p_0^{a_0} \cdot p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$$

with  $a_k \geq 1$ . This guarantees that the mapping  $\langle \rangle(a_0, \dots, a_k) = \langle a_0, \dots, a_k \rangle$  is injective: different sequences are mapped to different numbers; to each number only at most one sequence corresponds.

We'll now show that the operations of determining the length of a sequence, determining its  $i$ th element, appending an element to a sequence, and concatenating two sequences, are all primitive recursive.

**Proposition rec.1.** *The function  $\text{len}(s)$ , which returns the length of the sequence  $s$ , is primitive recursive.*

*Proof.* Let  $R(i, s)$  be the relation defined by

$$R(i, s) \text{ iff } p_i \mid s \wedge p_{i+1} \nmid s.$$

$R$  is clearly primitive recursive. Whenever  $s$  is the code of a non-empty sequence, i.e.,

$$s = p_0^{a_0+1} \cdot \dots \cdot p_k^{a_k+1},$$

$R(i, s)$  holds if  $p_i$  is the largest prime such that  $p_i \mid s$ , i.e.,  $i = k$ . The length of  $s$  thus is  $i + 1$  iff  $p_i$  is the largest prime that divides  $s$ , so we can let

$$\text{len}(s) = \begin{cases} 0 & \text{if } s = 0 \text{ or } s = 1 \\ 1 + (\min i < s) R(i, s) & \text{otherwise} \end{cases}$$

We can use bounded minimization, since there is only one  $i$  that satisfies  $R(s, i)$  when  $s$  is a code of a sequence, and if  $i$  exists it is less than  $s$  itself.  $\square$

**Proposition rec.2.** *The function  $\text{append}(s, a)$ , which returns the result of appending  $a$  to the sequence  $s$ , is primitive recursive.*

*Proof.* append can be defined by:

$$\text{append}(s, a) = \begin{cases} 2^{a+1} & \text{if } s = 0 \text{ or } s = 1 \\ s \cdot p_{\text{len}(s)}^{a+1} & \text{otherwise.} \end{cases}$$

□

**Proposition rec.3.** *The function  $\text{element}(s, i)$ , which returns the  $i$ th element of  $s$  (where the initial element is called the 0th), or 0 if  $i$  is greater than or equal to the length of  $s$ , is primitive recursive.*

*Proof.* Note that  $a$  is the  $i$ th element of  $s$  iff  $p_i^{a+1}$  is the largest power of  $p_i$  that divides  $s$ , i.e.,  $p_i^{a+1} \mid s$  but  $p_i^{a+2} \nmid s$ . So:

$$\text{element}(s, i) = \begin{cases} 0 & \text{if } i \geq \text{len}(s) \\ (\min a < s) (p_i^{a+2} \nmid s) & \text{otherwise.} \end{cases}$$

□

Instead of using the official names for the functions defined above, we introduce a more compact notation. We will use  $(s)_i$  instead of  $\text{element}(s, i)$ , and  $\langle s_0, \dots, s_k \rangle$  to abbreviate

$$\text{append}(\text{append}(\dots \text{append}(A, s_0) \dots), s_k).$$

Note that if  $s$  has length  $k$ , the elements of  $s$  are  $(s)_0, \dots, (s)_{k-1}$ .

**Proposition rec.4.** *The function  $\text{concat}(s, t)$ , which concatenates two sequences, is primitive recursive.*

*Proof.* We want a function  $\text{concat}$  with the property that

$$\text{concat}(\langle a_0, \dots, a_k \rangle, \langle b_0, \dots, b_l \rangle) = \langle a_0, \dots, a_k, b_0, \dots, b_l \rangle.$$

We'll use a "helper" function  $\text{hconcat}(s, t, n)$  which concatenates the first  $n$  symbols of  $t$  to  $s$ . This function can be defined by primitive recursion as follows:

$$\begin{aligned} \text{hconcat}(s, t, 0) &= s \\ \text{hconcat}(s, t, n+1) &= \text{append}(\text{hconcat}(s, t, n), (t)_n) \end{aligned}$$

Then we can define  $\text{concat}$  by

$$\text{concat}(s, t) = \text{hconcat}(s, t, \text{len}(t)).$$

□

We will write  $s \frown t$  instead of  $\text{concat}(s, t)$ .

It will be useful for us to be able to bound the numeric code of a sequence in terms of its length and its largest element. Suppose  $s$  is a sequence of length  $k$ , each element of which is less than equal to some number  $x$ . Then  $s$  has at most  $k$  prime factors, each at most  $p_{k-1}$ , and each raised to at most  $x + 1$  in the prime factorization of  $s$ . In other words, if we define

$$\text{sequenceBound}(x, k) = p_{k-1}^{k \cdot (x+1)},$$

then the numeric code of the sequence  $s$  described above is at most  $\text{sequenceBound}(x, k)$ .

Having such a bound on sequences gives us a way of defining new functions using bounded search. For example, we can define  $\text{concat}$  using bounded search. All we need to do is write down a primitive recursive *specification* of the object (number of the concatenated sequence) we are looking for, and a bound on how far to look. The following works:

$$\begin{aligned} \text{concat}(s, t) = & (\min v < \text{sequenceBound}(s + t, \text{len}(s) + \text{len}(t))) \\ & (\text{len}(v) = \text{len}(s) + \text{len}(t) \wedge \\ & (\forall i < \text{len}(s)) ((v)_i = (s)_i) \wedge \\ & (\forall j < \text{len}(t)) ((v)_{\text{len}(s)+j} = (t)_j)) \end{aligned}$$

**Problem rec.1.** Show that there is a primitive recursive function  $\text{sconcat}(s)$  with the property that

$$\text{sconcat}(\langle s_0, \dots, s_k \rangle) = s_0 \frown \dots \frown s_k.$$

**Problem rec.2.** Show that there is a primitive recursive function  $\text{tail}(s)$  with the property that

$$\begin{aligned} \text{tail}(\Lambda) &= 0 \text{ and} \\ \text{tail}(\langle s_0, \dots, s_k \rangle) &= \langle s_1, \dots, s_k \rangle. \end{aligned}$$

cmp:rec:seq;  
prop:subseq **Proposition rec.5.** *The function  $\text{subseq}(s, i, n)$  which returns the subsequence of  $s$  of length  $n$  beginning at the  $i$ th element, is primitive recursive.*

*Proof.* Exercise. □

**Problem rec.3.** Prove [Proposition rec.5](#).

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## Bibliography