A characteristic of the natural numbers is that every natural number can be reached from 0 by applying the successor operation “+1” finitely many times—any natural number is either 0 or the successor of . . . the successor of 0.

One way to specify a function $f : \mathbb{N} \rightarrow \mathbb{N}$ that makes use of this fact is this: (a) specify what the value of $f$ is for argument 0, and (b) also specify how, given the value of $f(x)$, compute the value of $f(x + 1)$. For (a) tells us directly what $f(0)$ is, so $f$ is defined for 0. Now, using the instruction given by (b) for $x = 0$, we can compute $f(1) = f(0 + 1)$ from $f(0)$. Using the same instructions for $x = 1$, we compute $f(2) = f(1 + 1)$ from $f(1)$, and so on. For every natural number $x$, we’ll eventually reach the step where we define $f(x)$ from $f(x + 1)$, and so $f(x)$ is defined for all $x \in \mathbb{N}$.

For instance, suppose we specify $h : \mathbb{N} \rightarrow \mathbb{N}$ by the following two equations:

$$
\begin{align*}
h(0) &= 1 \\
h(x + 1) &= 2 \cdot h(x).
\end{align*}
$$

If we already know how to multiply, then these equations give us the information required for (a) and (b) above. Successively the second equation, we get that

$$
\begin{align*}
h(1) &= 2 \cdot h(0) = 2, \\
h(2) &= 2 \cdot h(1) = 2 \cdot 2, \\
h(3) &= 2 \cdot h(2) = 2 \cdot 2 \cdot 2, \\
&\quad \vdots
\end{align*}
$$

We see that the function $h$ we have specified is $h(x) = 2^x$.

The characteristic feature of the natural numbers guarantees that there is only one function $d$ that meets these two criteria. A pair of equations like these is called a definition by primitive recursion of the function $d$. It is so-called because we define $f$ “recursively,” i.e., the definition, specifically the second equation, involves $f$ itself on the right-hand-side. It is “primitive” because in defining $f(x + 1)$ we only use the value $f(x)$, i.e., the immediately preceding value. This is the simplest way of defining a function on $\mathbb{N}$ recursively.

We can define even more fundamental functions like addition and multiplication by primitive recursion. In these cases, however, the functions in question are 2-place. We fix one of the argument places, and use the other for the recursion. E.g, to define $\text{add}(x, y)$ we can fix $x$ and define the value first for $y = 0$ and then for $y + 1$ in terms of $y$. Since $x$ is fixed, it will appear on the left and on the right side of the defining equations.

$$
\begin{align*}
\text{add}(x, 0) &= x \\
\text{add}(x, y + 1) &= \text{add}(x, y) + 1
\end{align*}
$$

These equations specify the value of add for all $x$ and $y$. To find $\text{add}(2, 3)$, for instance, we apply the defining equations for $x = 2$, using the first to find
add(2, 0) = 2, then using the second to successively find add(2, 1) = 2 + 1 = 3, add(2, 2) = 3 + 1 = 4, add(2, 3) = 4 + 1 = 5.

In the definition of add we used + on the right-hand-side of the second equation, but only to add 1. In other words, we used the successor function succ(z) = z + 1 and applied it to the previous value add(x, y) to define add(x, y + 1). So we can think of the recursive definition as given in terms of a single function which we apply to the previous value. However, it doesn’t hurt—and sometimes is necessary—to allow the function to depend not just on the previous value but also on x and y. Consider:

\[
\begin{align*}
\text{mult}(x, 0) &= 0 \\
\text{mult}(x, y + 1) &= \text{add}(\text{mult}(x, y), x).
\end{align*}
\]

This is a primitive recursive definition of a function mult by applying the function add to both the preceding value mult(x, y) and the first argument x. It also defines the function mult(x, y) for all arguments x and y. For instance, mult(2, 3) is determined by successively computing mult(2, 0), mult(2, 1), mult(2, 2), and mult(2, 3):

\[
\begin{align*}
\text{mult}(2, 0) &= 0 \\
\text{mult}(2, 1) &= \text{mult}(2, 0 + 1) = \text{add}(\text{mult}(2, 0), 2) = \text{add}(0, 2) = 2 \\
\text{mult}(2, 2) &= \text{mult}(2, 1 + 1) = \text{add}(\text{mult}(2, 1), 2) = \text{add}(2, 2) = 4 \\
\text{mult}(2, 3) &= \text{mult}(2, 2 + 1) = \text{add}(\text{mult}(2, 2), 2) = \text{add}(4, 2) = 6.
\end{align*}
\]

The general pattern then is this: to give a primitive recursive definition of a function h(x0, . . . , xk, y), we provide two equations. The first defines the value of h(x0, . . . , xk, 0) without reference to f. The second defines the value of h(x0, . . . , xk, y + 1) in terms of h(x0, . . . , xk, y), the other arguments x0, . . . , xk, and y. Only the immediately preceding value of h may be used in that second equation. If we think of the operations given by the right-hand-sides of these two equations as themselves being functions f and g, then the pattern to define a new function h by primitive recursion is this:

\[
\begin{align*}
h(x_0, \ldots, x_k, 0) &= f(x_0, \ldots, x_k) \\
h(x_0, \ldots, x_k, y + 1) &= g(x_0, \ldots, x_k, y, h(x_0, \ldots, x_k, y)).
\end{align*}
\]

In the case of add, we have k = 0 and f(x0) = x0 (the identity function), and g(x0, y, z) = z + 1 (the 3-place function that returns the successor of its third argument):

\[
\begin{align*}
\text{add}(x_0, 0) &= f(x_0) = x_0 \\
\text{add}(x_0, y + 1) &= g(x_0, y, \text{add}(x_0, y)) = \text{succ}(\text{add}(x_0, y)).
\end{align*}
\]

In the case of mult, we have f(x0) = 0 (the constant function always returning 0) and g(x0, y, z) = add(z, x0) (the 3-place function that returns the sum
of its first and last argument):

\[
\begin{align*}
\text{mult}(x_0, 0) &= f(x_0) = 0 \\
\text{mult}(x_0, y + 1) &= g(x_0, y, \text{mult}(x_0, y)) = \text{add}(\text{mult}(x_0, y), x_0).
\end{align*}
\]

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Bibliography