

## rec.1 Primitive Recursion

cmp:rec:pre:  
sec Suppose we specify that a certain function  $l$  from  $\mathbb{N}$  to  $\mathbb{N}$  satisfies the following two clauses: explanation

$$\begin{aligned}l(0) &= 1 \\l(x+1) &= 2 \cdot l(x).\end{aligned}$$

It is pretty clear that there is only one function,  $l$ , that meets these two criteria. This is an instance of a *definition by primitive recursion*. We can define even more fundamental functions like addition and multiplication by

$$\begin{aligned}f(x, 0) &= x \\f(x, y+1) &= f(x, y) + 1\end{aligned}$$

and

$$\begin{aligned}g(x, 0) &= 0 \\g(x, y+1) &= f(g(x, y), x).\end{aligned}$$

Exponentiation can also be defined recursively, by

$$\begin{aligned}h(x, 0) &= 1 \\h(x, y+1) &= g(h(x, y), x).\end{aligned}$$

We can also compose functions to build more complex ones; for example,

$$\begin{aligned}k(x) &= x^x + (x+3) \cdot x \\&= f(h(x, x), g(f(x, 3), x)).\end{aligned}$$

Let  $\text{zero}(x)$  be the function that always returns 0, regardless of what  $x$  is, and let  $\text{succ}(x) = x+1$  be the successor function. The set of *primitive recursive functions* is the set of functions from  $\mathbb{N}^n$  to  $\mathbb{N}$  that you get if you start with zero and succ by iterating the two operations above, primitive recursion and composition. The idea is that primitive recursive functions are defined in a straightforward and explicit way, so that it is intuitively clear that each one can be computed using finite means.

**Definition rec.1.** If  $f$  is a  $k$ -place function and  $g_0, \dots, g_{k-1}$  are  $l$ -place functions on the natural numbers, the *composition* of  $f$  with  $g_0, \dots, g_{k-1}$  is the  $l$ -place function  $h$  defined by

$$h(x_0, \dots, x_{l-1}) = f(g_0(x_0, \dots, x_{l-1}), \dots, g_{k-1}(x_0, \dots, x_{l-1})).$$

**Definition rec.2.** If  $f$  is a  $k$ -place function and  $g$  is a  $(k+2)$ -place function, then the function defined by *primitive recursion from  $f$  and  $g$*  is the  $(k+1)$ -place function  $h$  defined by the equations

$$\begin{aligned}h(0, z_0, \dots, z_{k-1}) &= f(z_0, \dots, z_{k-1}) \\h(x+1, z_0, \dots, z_{k-1}) &= g(x, h(x, z_0, \dots, z_{k-1}), z_0, \dots, z_{k-1})\end{aligned}$$

In addition to zero and succ, we will include among primitive recursive functions the projection functions,

$$P_i^n(x_0, \dots, x_{n-1}) = x_i,$$

for each natural number  $n$  and  $i < n$ . These are not terribly exciting in themselves:  $P_i^n$  is simply the  $k$ -place function that always returns its  $i$ th argument. But they allow us to define new functions by disregarding arguments or switching arguments, as we'll see later.

In the end, we have the following:

**Definition rec.3.** The set of primitive recursive functions is the set of functions from  $\mathbb{N}^n$  to  $\mathbb{N}$ , defined inductively by the following clauses:

1. zero is primitive recursive.
2. succ is primitive recursive.
3. Each projection function  $P_i^n$  is primitive recursive.
4. If  $f$  is a  $k$ -place primitive recursive function and  $g_0, \dots, g_{k-1}$  are  $l$ -place primitive recursive functions, then the composition of  $f$  with  $g_0, \dots, g_{k-1}$  is primitive recursive.
5. If  $f$  is a  $k$ -place primitive recursive function and  $g$  is a  $k+2$ -place primitive recursive function, then the function defined by primitive recursion from  $f$  and  $g$  is primitive recursive.

explanation Put more concisely, the set of primitive recursive functions is the smallest set containing zero, succ, and the projection functions  $P_j^n$ , and which is closed under composition and primitive recursion.

Another way of describing the set of primitive recursive functions keeps track of the “stage” at which a function enters the set. Let  $S_0$  denote the set of starting functions: zero, succ, and the projections. Once  $S_i$  has been defined, let  $S_{i+1}$  be the set of all functions you get by applying a single instance of composition or primitive recursion to functions in  $S_i$ . Then

$$S = \bigcup_{i \in \mathbb{N}} S_i$$

is the set of all primitive recursive functions

Our definition of composition may seem too rigid, since  $g_0, \dots, g_{k-1}$  are all required to have the same arity  $l$ . (Remember that the *arity* of a function is the number of arguments; an  $l$ -place function has arity  $l$ .) But adding the projection functions provides the desired flexibility. For example, suppose  $f$  and  $g$  are 3-place functions and  $h$  is the 2-place function defined by

$$h(x, y) = f(x, g(x, x, y), y).$$

The definition of  $h$  can be rewritten with the projection functions, as

$$h(x, y) = f(P_0^2(x, y), g(P_0^2(x, y), P_0^2(x, y), P_1^2(x, y)), P_1^2(x, y)).$$

Then  $h$  is the composition of  $f$  with  $P_0^2$ ,  $l$ , and  $P_1^2$ , where

$$l(x, y) = g(P_0^2(x, y), P_0^2(x, y), P_1^2(x, y)),$$

i.e.,  $l$  is the composition of  $g$  with  $P_0^2$ ,  $P_0^2$ , and  $P_1^2$ .

For another example, let us again consider addition. This is described recursively by the following two equations:

$$\begin{aligned} x + 0 &= x \\ x + (y + 1) &= \text{succ}(x + y). \end{aligned}$$

In other words, addition is the function  $\text{add}$  defined recursively by the equations

$$\begin{aligned} \text{add}(0, x) &= x \\ \text{add}(y + 1, x) &= \text{succ}(\text{add}(y, x)). \end{aligned}$$

But even this is not a strict primitive recursive definition; we need to put it in the form

$$\begin{aligned} \text{add}(0, x) &= f(x) \\ \text{add}(y + 1, x) &= g(y, \text{add}(y, x), x) \end{aligned}$$

for some 1-place primitive recursive function  $f$  and some 3-place primitive recursive function  $g$ . We can take  $f$  to be  $P_0^1$ , and we can define  $g$  using composition,

$$g(y, w, x) = \text{succ}(P_1^3(y, w, x)).$$

The function  $g$ , being the composition of basic primitive recursive functions, is primitive recursive; and hence so is  $h$ . (Note that, strictly speaking, we have defined the function  $g(y, x)$  meeting the recursive specification of  $x + y$ ; in other words, the variables are in a different order. Luckily, addition is commutative, so here the difference is not important; otherwise, we could define the function  $g'$  by

$$g'(x, y) = g(P_1^2(y, x), P_0^2(y, x)) = g(y, x),$$

using composition.

One advantage to having the precise description of the primitive recursive functions is that we can be systematic in describing them. For example, we can assign a “notation” to each such function, as follows. Use symbols  $\text{zero}$ ,  $\text{succ}$ , and  $P_i^n$  for zero, successor, and the projections. Now suppose  $f$  is defined by composition from a  $k$ -place function  $h$  and  $l$ -place functions  $g_0, \dots, g_{k-1}$ , and we have assigned notations  $H, G_0, \dots, G_{k-1}$  to the latter functions. Then, using a new symbol  $\text{Comp}_{k,l}$ , we can denote the function  $f$  by

[explanation](#)

$\text{Comp}_{k,l}[H, G_0, \dots, G_{k-1}]$ . For the functions defined by primitive recursion, we can use analogous notations of the form  $\text{Rec}_k[G, H]$ , where  $k$  denotes that arity of the function being defined. With this setup, we can denote the addition function by

$$\text{Rec}_2[P_0^1, \text{Comp}_{1,3}[\text{succ}, P_1^3]].$$

Having these notations sometimes proves useful.

**Problem rec.1.** Multiplication satisfies the recursive equations

$$\begin{aligned} 0 \cdot y &= y \\ (x + 1) \cdot y &= (x \cdot y) + x \end{aligned}$$

Give the explicit precise definition of the function  $\text{mult}(x, y) = x \cdot y$ , assuming that  $\text{add}(x, y) = x + y$  is already defined. Give the complete notation for  $\text{mult}$ .

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## Bibliography