

## rec.1 Primitive Recursive Relations

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**Definition rec.1.** A relation  $R(\vec{x})$  is said to be primitive recursive if its characteristic function,

$$\chi_R(\vec{x}) = \begin{cases} 1 & \text{if } R(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

In other words, when one speaks of a primitive recursive relation  $R(\vec{x})$ , one is referring to a relation of the form  $\chi_R(\vec{x}) = 1$ , where  $\chi_R$  is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation  $\text{IsZero}(x)$ , which holds if and only if  $x = 0$ , corresponds to the function  $\chi_{\text{IsZero}}$ , defined using primitive recursion by

$$\chi_{\text{IsZero}}(0) = 1, \quad \chi_{\text{IsZero}}(x + 1) = 0.$$

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation,  $x = y$ , defined by  $\text{IsZero}(|x - y|)$
2. The less-than relation,  $x \leq y$ , defined by  $\text{IsZero}(x \dot{-} y)$

**Proposition rec.2.** *The set of primitive recursive relations is closed under boolean operations, that is, if  $P(\vec{x})$  and  $Q(\vec{x})$  are primitive, so are*

1.  $\neg R(\vec{x})$
2.  $P(\vec{x}) \wedge Q(\vec{x})$
3.  $P(\vec{x}) \vee Q(\vec{x})$
4.  $P(\vec{x}) \rightarrow Q(\vec{x})$

*Proof.* Suppose  $P(\vec{x})$  and  $Q(\vec{x})$  are primitive recursive, i.e., their characteristic functions  $\chi_P$  and  $\chi_Q$  are. We have to show that the characteristic functions of  $\neg R(\vec{x})$ , etc., are also primitive recursive.

$$\chi_{\neg P}(\vec{x}) = \begin{cases} 0 & \text{if } \chi_P(\vec{x}) = 1 \\ 1 & \text{otherwise} \end{cases}$$

We can define  $\chi_{\neg P}(\vec{x})$  as  $1 \dot{-} \chi_P(\vec{x})$ .

$$\chi_{P \wedge Q}(\vec{x}) = \begin{cases} 1 & \text{if } \chi_P(\vec{x}) = \chi_Q(\vec{x}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We can define  $\chi_{P \wedge Q}(\vec{x})$  as  $\chi_P(\vec{x}) \cdot \chi_Q(\vec{x})$  or as  $\min(\chi_P(\vec{x}), \chi_Q(\vec{x}))$ .

Similarly,  $\chi_{P \vee Q}(\vec{x}) = \max(\chi_P(\vec{x}), \chi_Q(\vec{x}))$  and  $\chi_{P \rightarrow Q}(\vec{x}) = \max(1 \dot{-} \chi_P(\vec{x}), \chi_Q(\vec{x}))$ .  $\square$

**Proposition rec.3.** *The set of primitive recursive relations is closed under bounded quantification, i.e., if  $R(\vec{x}, z)$  is a primitive recursive relation, then so are the relations  $(\forall z < y) R(\vec{x}, z)$  and  $(\exists z < y) R(\vec{x}, z)$ .*

*( $(\forall z < y) R(\vec{x}, z)$  holds of  $\vec{x}$  and  $y$  if and only if  $R(\vec{x}, z)$  holds for every  $z$  less than  $y$ , and similarly for  $(\exists z < y) R(\vec{x}, z)$ .)*

*Proof.* By convention, we take  $(\forall z < 0) R(\vec{x}, z)$  to be true (for the trivial reason that there are no  $z$  less than 0) and  $(\exists z < 0) R(\vec{x}, z)$  to be false. A universal quantifier functions just like a finite product or iterated minimum, i.e., if  $P(\vec{x}, y) \Leftrightarrow (\forall z < y) R(\vec{x}, z)$  then  $\chi_P(\vec{x}, y)$  can be defined by

$$\begin{aligned}\chi_P(\vec{x}, 0) &= 1 \\ \chi_P(\vec{x}, y + 1) &= \min(\chi_P(\vec{x}, y), \chi_R(\vec{x}, y + 1)).\end{aligned}$$

Bounded existential quantification can similarly be defined using max. Alternatively, it can be defined from bounded universal quantification, using the equivalence  $(\exists z < y) R(\vec{x}, z) \Leftrightarrow \neg(\forall z < y) \neg R(\vec{x}, z)$ . Note that, for example, a bounded quantifier of the form  $(\exists x \leq y) \dots x \dots$  is equivalent to  $(\exists x < y + 1) \dots x \dots$ .  $\square$

Another useful primitive recursive function is the conditional function,  $\text{cond}(x, y, z)$ , defined by

$$\text{cond}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}$$

This is defined recursively by

$$\text{cond}(0, y, z) = y, \quad \text{cond}(x + 1, y, z) = z.$$

One can use this to justify definitions of primitive recursive functions by cases from primitive recursive relations:

**Proposition rec.4.** *If  $g_0(\vec{x}), \dots, g_m(\vec{x})$  are functions, and  $R_1(\vec{x}), \dots, R_{m-1}(\vec{x})$  are primitive recursive relations, then the function  $f$  defined by*

$$f(\vec{x}) = \begin{cases} g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\ g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\ \vdots & \\ g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\ g_m(\vec{x}) & \text{otherwise} \end{cases}$$

*is also primitive recursive.*

*Proof.* When  $m = 1$ , this is just the function defined by

$$f(\vec{x}) = \text{cond}(\chi_{\neg R_0}(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For  $m$  greater than 1, one can just compose definitions of this form.  $\square$

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**Bibliography**