Definition rec.1. A relation $R(\vec{x})$ is said to be primitive recursive if its characteristic function,
\[ \chi_R(\vec{x}) = \begin{cases} 1 & \text{if } R(\vec{x}) \\ 0 & \text{otherwise} \end{cases} \]
is primitive recursive.

In other words, when one speaks of a primitive recursive relation $R(\vec{x})$, one is referring to a relation of the form $\chi_R(\vec{x}) = 1$, where $\chi_R$ is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation $\text{IsZero}(x)$, which holds if and only if $x = 0$, corresponds to the function $\chi_{\text{IsZero}}$, defined using primitive recursion by
\[ \chi_{\text{IsZero}}(0) = 1, \quad \chi_{\text{IsZero}}(x + 1) = 0. \]

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation, $x = y$, defined by $\text{IsZero}(|x - y|)$
2. The less-than relation, $x \leq y$, defined by $\text{IsZero}(x - y)$

**Proposition rec.2.** The set of primitive recursive relations is closed under boolean operations, that is, if $P(\vec{x})$ and $Q(\vec{x})$ are primitive recursive, so are

1. $\neg P(\vec{x})$
2. $P(\vec{x}) \land Q(\vec{x})$
3. $P(\vec{x}) \lor Q(\vec{x})$
4. $P(\vec{x}) \rightarrow Q(\vec{x})$

**Proof.** Suppose $P(\vec{x})$ and $Q(\vec{x})$ are primitive recursive, i.e., their characteristic functions $\chi_P$ and $\chi_Q$ are. We have to show that the characteristic functions of $\neg P(\vec{x})$, etc., are also primitive recursive.

\[ \chi_{\neg P}(\vec{x}) = \begin{cases} 0 & \text{if } \chi_P(\vec{x}) = 1 \\ 1 & \text{otherwise} \end{cases} \]

We can define $\chi_{\neg P}(\vec{x})$ as $1 - \chi_P(\vec{x})$.

\[ \chi_{P \land Q}(\vec{x}) = \begin{cases} 1 & \text{if } \chi_P(\vec{x}) = \chi_Q(\vec{x}) = 1 \\ 0 & \text{otherwise} \end{cases} \]

We can define $\chi_{P \land Q}(\vec{x})$ as $\chi_P(\vec{x}) \cdot \chi_Q(\vec{x})$ or as $\min(\chi_P(\vec{x}), \chi_Q(\vec{x}))$.

Similarly, $\chi_{P \lor Q}(\vec{x}) = \max(\chi_P(\vec{x}), \chi_Q(\vec{x}))$ and $\chi_{P \rightarrow Q}(\vec{x}) = \max(1 - \chi_P(\vec{x}), \chi_Q(\vec{x})).$
**Proposition rec.3.** The set of primitive recursive relations is closed under bounded quantification, i.e., if \( R(\vec{x}, z) \) is a primitive recursive relation, then so are the relations \( (\forall z < y) R(\vec{x}, z) \) and \( (\exists z < y) R(\vec{x}, z) \).

\((\forall z < y) R(\vec{x}, z) \) holds of \( \vec{x} \) and \( y \) if and only if \( R(\vec{x}, z) \) holds for every \( z \) less than \( y \), and similarly for \( (\exists z < y) R(\vec{x}, z) \).

**Proof.** By convention, we take \((\forall z < 0) R(\vec{x}, z)\) to be true (for the trivial reason that there are no \( z \) less than 0) and \((\exists z < 0) R(\vec{x}, z)\) to be false. A universal quantifier functions just like a finite product or iterated minimum, i.e., if \( P(\vec{x}, y) \iff (\forall z < y) R(\vec{x}, z) \) then \( \chi_P(\vec{x}, y) \) can be defined by

\[
\chi_P(\vec{x}, 0) = 1 \\
\chi_P(\vec{x}, y + 1) = \min(\chi_P(\vec{x}, y), \chi_R(\vec{x}, y)).
\]

Bounded existential quantification can similarly be defined using max. Alternatively, it can be defined from bounded universal quantification, using the equivalence \((\exists z < y) R(\vec{x}, z) \iff \neg(\forall z < y) \neg R(\vec{x}, z)\). Note that, for example, a bounded quantifier of the form \((\exists x \leq y) \ldots x \ldots \) is equivalent to \((\exists x < y + 1) \ldots x \ldots \).

Another useful primitive recursive function is the conditional function, \( \text{cond}(x, y, z) \), defined by

\[
\text{cond}(x, y, z) = \begin{cases} 
  y & \text{if } x = 0 \\
  z & \text{otherwise}.
\end{cases}
\]

This is defined recursively by

\[
\text{cond}(0, y, z) = y, \quad \text{cond}(x + 1, y, z) = z.
\]

One can use this to justify definitions of primitive recursive functions by cases from primitive recursive relations:

**Proposition rec.4.** If \( g_0(\vec{x}), \ldots, g_m(\vec{x}) \) are primitive recursive functions, and \( R_0(\vec{x}), \ldots, R_{m-1}(\vec{x}) \) are primitive recursive relations, then the function \( f \) defined by

\[
f(\vec{x}) = \begin{cases} 
  g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\
  g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\
  \vdots \\
  g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\
  g_m(\vec{x}) & \text{otherwise}
\end{cases}
\]

is also primitive recursive.
Proof. When $m = 1$, this is just the function defined by

$$f(\vec{x}) = \text{cond}(\chi_{\sim R_0}(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For $m$ greater than 1, one can just compose definitions of this form. □

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Bibliography