

## rec.1 Primitive Recursive Relations

cmp:rec:pr:  
sec

**Definition rec.1.** A relation  $R(\vec{x})$  is said to be primitive recursive if its characteristic function,

$$\chi_R(\vec{x}) = \begin{cases} 1 & \text{if } R(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

In other words, when one speaks of a primitive recursive relation  $R(\vec{x})$ , one is referring to a relation of the form  $\chi_R(\vec{x}) = 1$ , where  $\chi_R$  is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation  $\text{IsZero}(x)$ , which holds if and only if  $x = 0$ , corresponds to the function  $\chi_{\text{IsZero}}$ , defined using primitive recursion by

$$\chi_{\text{IsZero}}(0) = 1, \quad \chi_{\text{IsZero}}(x + 1) = 0.$$

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation,  $x = y$ , defined by  $\text{IsZero}(|x - y|)$
2. The less-than relation,  $x \leq y$ , defined by  $\text{IsZero}(x \dot{-} y)$

Furthermore, the set of primitive recursive relations is closed under boolean operations:

1. Negation,  $\neg P$
2. Conjunction,  $P \wedge Q$
3. Disjunction,  $P \vee Q$
4. If ... then,  $P \rightarrow Q$

are all primitive recursive, if  $P$  and  $Q$  are. For suppose  $\chi_P(\vec{z})$  and  $\chi_Q(\vec{z})$  are primitive recursive. Then the relation  $R(\vec{z})$  that holds iff both  $P(\vec{z})$  and  $Q(\vec{z})$  hold has the characteristic function  $\chi_R(\vec{z}) = \text{and}(\chi_P(\vec{z}), \chi_Q(\vec{z}))$ .

One can also define relations using bounded quantification:

1. Bounded universal quantification: if  $R(x, \vec{z})$  is a primitive recursive relation, then so is the relation

$$(\forall x < y) R(x, \vec{z})$$

which holds if and only if  $R(x, \vec{z})$  holds for every  $x$  less than  $y$ .

2. Bounded existential quantification: if  $R(x, \vec{z})$  is a primitive recursive relation, then so is

$$(\exists x < y) R(x, \vec{z}).$$

By convention, we take  $(\forall x < 0) R(x, \vec{z})$  to be true (for the trivial reason that there *are* no  $x$  less than 0) and  $(\exists x < 0) R(x, \vec{z})$  to be false. A universal quantifier functions just like a finite product; it can also be defined directly by

$$g(0, \vec{z}) = 1, \quad g(y + 1, \vec{z}) = \text{and}(g(y, \vec{z}), \chi_R(y, \vec{z})).$$

Bounded existential quantification can similarly be defined using *or*. Alternatively, it can be defined from bounded universal quantification, using the equivalence,  $(\exists x < y) \varphi(x) \leftrightarrow \neg(\forall x < y) \neg\varphi(x)$ . Note that, for example, a bounded quantifier of the form  $(\exists x \leq y) \dots x \dots$  is equivalent to  $(\exists x < y + 1) \dots x \dots$ .

Another useful primitive recursive function is:

1. The conditional function,  $\text{cond}(x, y, z)$ , defined by

$$\text{cond}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise} \end{cases}$$

This is defined recursively by

$$\text{cond}(0, y, z) = y, \quad \text{cond}(x + 1, y, z) = z.$$

One can use this to justify:

1. Definition by cases: if  $g_0(\vec{x}), \dots, g_m(\vec{x})$  are functions, and  $R_1(\vec{x}), \dots, R_{m-1}(\vec{x})$  are relations, then the function  $f$  defined by

$$f(\vec{x}) = \begin{cases} g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\ g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\ \vdots & \\ g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\ g_m(\vec{x}) & \text{otherwise} \end{cases}$$

is also primitive recursive.

When  $m = 1$ , this is just the function defined by

$$f(\vec{x}) = \text{cond}(\chi_{\neg R_0}(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For  $m$  greater than 1, one can just compose definitions of this form.

## Photo Credits

## Bibliography