rec.1 Primitive Recursive Relations

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Definition rec.1. A relation $R(\vec{x})$ is said to be primitive recursive if its characteristic function,

$$\chi_R(\vec{x}) = \begin{cases} 1 & \text{if } R(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

In other words, when one speaks of a primitive recursive relation $R(\vec{x})$, one is referring to a relation of the form $\chi_R(\vec{x}) = 1$, where χ_R is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation $\mathrm{IsZero}(x)$, which holds if and only if $x = 0$, corresponds to the function χ_{IsZero}, defined using primitive recursion by

$$\chi_{\mathrm{IsZero}}(0) = 1, \quad \chi_{\mathrm{IsZero}}(x+1) = 0.$$

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation, $x = y$, defined by $\mathrm{IsZero}(|x - y|)$
2. The less-than relation, $x \leq y$, defined by $\mathrm{IsZero}(x \dot{-} y)$

Furthermore, the set of primitive recursive relations is closed under boolean operations:

1. Negation, $\neg P$
2. Conjunction, $P \land Q$
3. Disjunction, $P \lor Q$
4. If ... then, $P \to Q$

are all primitive recursive, if P and Q are. For suppose $\chi_P(\vec{z})$ an $\chi_Q(\vec{z})$ are primitive recursive. Then the relation $R(\vec{z})$ that holds iff both $P(\vec{z})$ and $Q(\vec{z})$ hold has the characteristic function $\chi_R(\vec{z}) = \mathrm{and}(\chi_P(\vec{z}), \chi_Q(\vec{z}))$.

One can also define relations using bounded quantification:

1. Bounded universal quantification: if $R(x, \vec{z})$ is a primitive recursive relation, then so is the relation

   $$(\forall x < y)\; R(x, \vec{z})$$

   which holds if and only if $R(x, \vec{z})$ holds for every x less than y.

2. Bounded existential quantification: if $R(x, \vec{z})$ is a primitive recursive relation, then so is

   $$(\exists x < y)\; R(x, \vec{z}).$$

By convention, we take $(\forall x < 0)\ R(x, \vec{z})$ to be true (for the trivial reason that there *are* no x less than 0) and $(\exists x < 0)\ R(x, \vec{z})$ to be false. A universal quantifier functions just like a finite product; it can also be defined directly by

$$g(0, \vec{z}) = 1, \quad g(y+1, \vec{z}) = \mathrm{and}(g(y, \vec{z}), \chi_R(y, \vec{z})).$$

Bounded existential quantification can similarly be defined using or. Alternatively, it can be defined from bounded universal quantification, using the equivalence, $(\exists x < y)\ \varphi(x) \leftrightarrow \neg(\forall x < y)\ \neg\varphi(x)$. Note that, for example, a bounded quantifier of the form $(\exists x \leq y)\ \ldots x \ldots$ is equivalent to $(\exists x < y+1)\ \ldots x \ldots$.

Another useful primitive recursive function is:

1. The conditional function, $\mathrm{cond}(x, y, z)$, defined by

$$\mathrm{cond}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise} \end{cases}$$

This is defined recursively by

$$\mathrm{cond}(0, y, z) = y, \quad \mathrm{cond}(x+1, y, z) = z.$$

One can use this to justify:

1. Definition by cases: if $g_0(\vec{x}), \ldots, g_m(\vec{x})$ are functions, and $R_1(\vec{x}), \ldots, R_{m-1}(\vec{x})$ are relations, then the function f defined by

$$f(\vec{x}) = \begin{cases} g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\ g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\ \vdots & \\ g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\ g_m(\vec{x}) & \text{otherwise} \end{cases}$$

   is also primitive recursive.

When $m = 1$, this is just the function defined by

$$f(\vec{x}) = \mathrm{cond}(\chi_{\neg R_0}(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For m greater than 1, one can just compose definitions of this form.

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Bibliography