Definition rec.1. A relation \( R(\vec{x}) \) is said to be primitive recursive if its characteristic function,
\[
\chi_R(\vec{x}) = \begin{cases} 
1 & \text{if } R(\vec{x}) \\
0 & \text{otherwise}
\end{cases}
\]
is primitive recursive.

In other words, when one speaks of a primitive recursive relation \( R(\vec{x}) \), one is referring to a relation of the form \( \chi_R(\vec{x}) = 1 \), where \( \chi_R \) is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation \( \text{IsZero}(x) \), which holds if and only if \( x = 0 \), corresponds to the function \( \chi_{\text{IsZero}} \), defined using primitive recursion by
\[
\chi_{\text{IsZero}}(0) = 1, \quad \chi_{\text{IsZero}}(x + 1) = 0.
\]

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation, \( x = y \), defined by \( \text{IsZero}(|x - y|) \)
2. The less-than relation, \( x \leq y \), defined by \( \text{IsZero}(x - y) \)

Proposition rec.2. The set of primitive recursive relations is closed under boolean operations, that is, if \( P(\vec{x}) \) and \( Q(\vec{x}) \) are primitive, so are

1. \( \neg R(\vec{x}) \)
2. \( P(\vec{x}) \wedge Q(\vec{x}) \)
3. \( P(\vec{x}) \lor Q(\vec{x}) \)
4. \( P(\vec{x}) \rightarrow Q(\vec{x}) \)

Proof. Suppose \( P(\vec{x}) \) and \( Q(\vec{x}) \) are primitive recursive, i.e., their characteristic functions \( \chi_P \) and \( \chi_Q \) are. We have to show that the characteristic functions of \( \neg R(\vec{x}) \), etc., are also primitive recursive.

\[
\chi_{\neg P}(\vec{x}) = \begin{cases} 
0 & \text{if } \chi_P(\vec{x}) = 1 \\
1 & \text{otherwise}
\end{cases}
\]

We can define \( \chi_{\neg P}(\vec{x}) \) as \( 1 - \chi_P(\vec{x}) \).

\[
\chi_{P \wedge Q}(\vec{x}) = \begin{cases} 
1 & \text{if } \chi_P(\vec{x}) = \chi_Q(\vec{x}) = 1 \\
0 & \text{otherwise}
\end{cases}
\]

We can define \( \chi_{P \wedge Q}(\vec{x}) \) as \( \chi_P(\vec{x}) \cdot \chi_Q(\vec{x}) \) or as \( \min(\chi_P(\vec{x}), \chi_Q(\vec{x})) \).

Similarly, \( \chi_{P \lor Q}(\vec{x}) = \max(\chi_P(\vec{x}), \chi_Q(\vec{x})) \) and \( \chi_{P \lor Q}(\vec{x}) = \max(1 - \chi_P(\vec{x}), \chi_Q(\vec{x})) \).
**Proposition rec.3.** The set of primitive recursive relations is closed under bounded quantification, i.e., if $R(\vec{x}, z)$ is a primitive recursive relation, then so are the relations $(\forall z < y) R(\vec{x}, z)$ and $(\exists z < y) R(\vec{x}, z)$.

$(\forall z < y) R(\vec{x}, z)$ holds of $\vec{x}$ and $y$ if and only if $R(\vec{x}, z)$ holds for every $z$ less than $y$, and similarly for $(\exists z < y) R(\vec{x}, z)$.

**Proof.** By convention, we take $(\forall z < 0) R(\vec{x}, z)$ to be true (for the trivial reason that there are no $z$ less than 0) and $(\exists z < 0) R(\vec{x}, z)$ to be false. A universal quantifier functions just like a finite product or iterated minimum, i.e., if $P(\vec{x}, y) \iff (\forall z < y) R(\vec{x}, z)$ then $\chi P(\vec{x}, y)$ can be defined by

$$\chi P(\vec{x}, 0) = 1$$

$$\chi P(\vec{x}, y + 1) = \min(\chi P(\vec{x}, y), \chi R(\vec{x}, y + 1))$$

Bounded existential quantification can similarly be defined using max. Alternatively, it can be defined from bounded universal quantification, using the equivalence $(\exists z < y) R(\vec{x}, z) \iff \neg (\forall z < y) \neg R(\vec{x}, z)$. Note that, for example, a bounded quantifier of the form $(\exists x \leq y) \ldots x \ldots$ is equivalent to $(\exists x < y + 1) \ldots x \ldots$.

Another useful primitive recursive function is the conditional function, $\text{cond}(x, y, z)$, defined by

$$\text{cond}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}$$

This is defined recursively by

$$\text{cond}(0, y, z) = y, \quad \text{cond}(x + 1, y, z) = z.$$ 

One can use this to justify definitions of primitive recursive functions by cases from primitive recursive relations:

**Proposition rec.4.** If $g_0(\vec{x})$, $\ldots$, $g_m(\vec{x})$ are functions, and $R_1(\vec{x}), \ldots, R_{m-1}(\vec{x})$ are primitive recursive relations, then the function $f$ defined by

$$f(\vec{x}) = \begin{cases} g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\ g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\ \vdots \\ g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\ g_m(\vec{x}) & \text{otherwise} \end{cases}$$

is also primitive recursive.

**Proof.** When $m = 1$, this is just the function defined by

$$f(\vec{x}) = \text{cond}(\chi \neg R_0(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For $m$ greater than 1, one can just compose definitions of this form. 

\[ \Box \]