rec.1 Primitive Recursive Relations

cmp:rec:prr:sec

Definition rec.1. A relation $R(\vec{x})$ is said to be primitive recursive if its characteristic function,

$$\chi_R(\vec{x}) = \begin{cases} 1 & \text{if } R(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

In other words, when one speaks of a primitive recursive relation $R(\vec{x})$, one is referring to a relation of the form $\chi_R(\vec{x}) = 1$, where χ_R is a primitive recursive function which, on any input, returns either 1 or 0. For example, the relation $\mathrm{IsZero}(x)$, which holds if and only if $x = 0$, corresponds to the function χ_{IsZero}, defined using primitive recursion by

$$\begin{aligned} \chi_{\mathrm{IsZero}}(0) &= 1, \\ \chi_{\mathrm{IsZero}}(x+1) &= 0. \end{aligned}$$

It should be clear that one can compose relations with other primitive recursive functions. So the following are also primitive recursive:

1. The equality relation, $x = y$, defined by $\mathrm{IsZero}(|x - y|)$
2. The less-than relation, $x \leq y$, defined by $\mathrm{IsZero}(x \dot{-} y)$

Proposition rec.2. *The set of primitive recursive relations is closed under Boolean operations, that is, if $P(\vec{x})$ and $Q(\vec{x})$ are primitive recursive, so are*

1. $\lnot P(\vec{x})$
2. $P(\vec{x}) \land Q(\vec{x})$
3. $P(\vec{x}) \lor Q(\vec{x})$
4. $P(\vec{x}) \rightarrow Q(\vec{x})$

Proof. Suppose $P(\vec{x})$ and $Q(\vec{x})$ are primitive recursive, i.e., their characteristic functions χ_P and χ_Q are. We have to show that the characteristic functions of $\lnot P(\vec{x})$, etc., are also primitive recursive.

$$\chi_{\lnot P}(\vec{x}) = \begin{cases} 0 & \text{if } \chi_P(\vec{x}) = 1 \\ 1 & \text{otherwise} \end{cases}$$

We can define $\chi_{\lnot P}(\vec{x})$ as $1 \dot{-} \chi_P(\vec{x})$.

$$\chi_{P \land Q}(\vec{x}) = \begin{cases} 1 & \text{if } \chi_P(\vec{x}) = \chi_Q(\vec{x}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We can define $\chi_{P \land Q}(\vec{x})$ as $\chi_P(\vec{x}) \cdot \chi_Q(\vec{x})$ or as $\min(\chi_P(\vec{x}), \chi_Q(\vec{x}))$. Similarly,

$$\begin{aligned} \chi_{P \lor Q}(\vec{x}) &= \max(\chi_P(\vec{x}), \chi_Q(\vec{x})) \text{ and} \\ \chi_{P \rightarrow Q}(\vec{x}) &= \max(1 \dot{-} \chi_P(\vec{x}), \chi_Q(\vec{x})). \end{aligned}$$ □

Proposition rec.3. *The set of primitive recursive relations is closed under bounded quantification, i.e., if $R(\vec{x}, z)$ is a primitive recursive relation, then so are the relations*

$$
\begin{aligned}
&(\forall z < y)\; R(\vec{x}, z) \text{ and} \\
&(\exists z < y)\; R(\vec{x}, z).
\end{aligned}
$$

$(\forall z < y)\; R(\vec{x}, z)$ holds of $\vec{x}$ and y if and only if $R(\vec{x}, z)$ holds for every z less than y, and similarly for $(\exists z < y)\; R(\vec{x}, z)$.

Proof. By convention, we take $(\forall z < 0)\; R(\vec{x}, z)$ to be true (for the trivial reason that there are no z less than 0) and $(\exists z < 0)\; R(\vec{x}, z)$ to be false. A bounded universal quantifier functions just like a finite product or iterated minimum, i.e., if $P(\vec{x}, y) \Leftrightarrow (\forall z < y)\; R(\vec{x}, z)$ then $\chi_P(\vec{x}, y)$ can be defined by

$$
\begin{aligned}
\chi_P(\vec{x}, 0) &= 1 \\
\chi_P(\vec{x}, y + 1) &= \min(\chi_P(\vec{x}, y), \chi_R(\vec{x}, y))).
\end{aligned}
$$

Bounded existential quantification can similarly be defined using max. Alternatively, it can be defined from bounded universal quantification, using the equivalence $(\exists z < y)\; R(\vec{x}, z) \leftrightarrow \neg(\forall z < y)\; \neg R(\vec{x}, z)$. Note that, for example, a bounded quantifier of the form $(\exists x \leq y)\; \dots x \dots$ is equivalent to $(\exists x < y + 1)\; \dots x \dots$. □

Problem rec.1. Show that the three place relation $x \equiv y \mod n$ (congruence modulo n) is primitive recursive.

Another useful primitive recursive function is the conditional function, $\mathrm{cond}(x, y, z)$, defined by

$$
\mathrm{cond}(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}
$$

This is defined recursively by

$$
\begin{aligned}
\mathrm{cond}(0, y, z) &= y, \\
\mathrm{cond}(x + 1, y, z) &= z.
\end{aligned}
$$

One can use this to justify definitions of primitive recursive functions by cases from primitive recursive relations:

Proposition rec.4. *If $g_0(\vec{x})$, …, $g_m(\vec{x})$ are primitive recursive functions, and $R_0(\vec{x})$, …, $R_{m-1}(\vec{x})$ are primitive recursive relations, then the function f defined by*

$$
f(\vec{x}) = \begin{cases} g_0(\vec{x}) & \text{if } R_0(\vec{x}) \\ g_1(\vec{x}) & \text{if } R_1(\vec{x}) \text{ and not } R_0(\vec{x}) \\ \vdots & \\ g_{m-1}(\vec{x}) & \text{if } R_{m-1}(\vec{x}) \text{ and none of the previous hold} \\ g_m(\vec{x}) & \text{otherwise} \end{cases}
$$

is also primitive recursive.

Proof. When $m = 1$, this is just the function defined by

$$f(\vec{x}) = \text{cond}(\chi_{\neg R_0}(\vec{x}), g_0(\vec{x}), g_1(\vec{x})).$$

For m greater than 1, one can just compose definitions of this form. $\square$

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Bibliography