

## rec.1 Primitive Recursion Notations

cmp:rec:not:  
sec

One advantage to having the precise inductive description of the primitive recursive functions is that we can be systematic in describing them. For example, we can assign a “notation” to each such function, as follows. Use symbols zero, succ, and  $P_i^n$  for zero, successor, and the projections. Now suppose  $h$  is defined by composition from a  $k$ -place function  $f$  and  $n$ -place functions  $g_0, \dots, g_{k-1}$ , and we have assigned notations  $F, G_0, \dots, G_{k-1}$  to the latter functions. Then, using a new symbol  $\text{Comp}_{k,n}$ , we can denote the function  $h$  by  $\text{Comp}_{k,n}[F, G_0, \dots, G_{k-1}]$ .

For functions defined by primitive recursion, we can use analogous notations. Suppose the  $(k+1)$ -ary function  $h$  is defined by primitive recursion from the  $k$ -ary function  $f$  and the  $(k+2)$ -ary function  $g$ , and the notations assigned to  $f$  and  $g$  are  $F$  and  $G$ , respectively. Then the notation assigned to  $h$  is  $\text{Rec}_k[F, G]$ .

Recall that the addition function is defined by primitive recursion as

$$\begin{aligned}\text{add}(x_0, 0) &= P_0^1(x_0) = x_0 \\ \text{add}(x_0, y + 1) &= \text{succ}(P_2^3(x_0, y, \text{add}(x_0, y))) = \text{add}(x_0, y) + 1\end{aligned}$$

Here the role of  $f$  is played by  $P_0^1$ , and the role of  $g$  is played by  $\text{succ}(P_2^3(x_0, y, z))$ , which is assigned the notation  $\text{Comp}_{1,3}[\text{succ}, P_2^3]$  as it is the result of defining a function by composition from the 1-ary function  $\text{succ}$  and the 3-ary function  $P_2^3$ . With this setup, we can denote the addition function by

$$\text{Rec}_1[P_0^1, \text{Comp}_{1,3}[\text{succ}, P_2^3]].$$

Having these notations sometimes proves useful, e.g., when enumerating primitive recursive functions.

**Problem rec.1.** Give the complete primitive recursive notation for mult.

## Photo Credits

## Bibliography