rec.1 Primitive Recursion Notations

cmp:rec:not:sec One advantage to having the precise inductive description of the primitive recursive functions is that we can be systematic in describing them. For example, we can assign a "notation" to each such function, as follows. Use symbols zero, succ, and P^n_i for zero, successor, and the projections. Now suppose h is defined by composition from a k-place function f and n-place functions g_0, …, g_{k-1}, and we have assigned notations F, G_0, …, G_{k-1} to the latter functions. Then, using a new symbol $\mathrm{Comp}_{k,n}$, we can denote the function h by $\mathrm{Comp}_{k,n}[F, G_0, \dots, G_{k-1}]$.

For functions defined by primitive recursion, we can use analogous notations. Suppose the $(k+1)$-ary function h is defined by primitive recursion from the k-ary function f and the $(k+2)$-ary function g, and the notations assigned to f and g are F and G, respectively. Then the notation assigned to h is $\mathrm{Rec}_k[F, G]$.

Recall that the addition function is defined by primitive recursion as

$$\begin{aligned} \mathrm{add}(x_0, 0) &= P^1_0(x_0) = x_0 \\ \mathrm{add}(x_0, y+1) &= \mathrm{succ}(P^3_2(x_0, y, \mathrm{add}(x_0, y))) = \mathrm{add}(x_0, y) + 1 \end{aligned}$$

Here the role of f is played by P^1_0, and the role of g is played by $\mathrm{succ}(P^3_2(x_0, y, z))$, which is assigned the notation $\mathrm{Comp}_{1,3}[\mathrm{succ}, P^3_2]$ as it is the result of defining a function by composition from the 1-ary function succ and the 3-ary function P^3_2. With this setup, we can denote the addition function by

$$\mathrm{Rec}_1[P^1_0, \mathrm{Comp}_{1,3}[\mathrm{succ}, P^3_2]].$$

Having these notations sometimes proves useful, e.g., when enumerating primitive recursive functions.

Problem rec.1. Give the complete primitive recursive notation for mult.

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Bibliography