

## rec.1 The Normal Form Theorem

cmp:rec:nft:  
sec  
cmp:rec:nft:  
thm:kleene-nf

**Theorem rec.1 (Kleene’s Normal Form Theorem).** *There is a primitive recursive relation  $T(e, x, s)$  and a primitive recursive function  $U(s)$ , with the following property: if  $f$  is any partial recursive function, then for some  $e$ ,*

$$f(x) \simeq U(\mu s T(e, x, s))$$

for every  $x$ .

The proof of the normal form theorem is involved, but the basic idea is [explanation](#) simple. Every partial recursive function has an *index*  $e$ , intuitively, a number coding its program or definition. If  $f(x) \downarrow$ , the computation can be recorded systematically and coded by some number  $s$ , and the fact that  $s$  codes the computation of  $f$  on input  $x$  can be checked primitive recursively using only  $x$  and the definition  $e$ . Consequently, the relation  $T$ , “the function with index  $e$  has a computation for input  $x$ , and  $s$  codes this computation,” is primitive recursive. Given the full record of the computation  $s$ , the “upshot” of  $s$  is the value of  $f(x)$ , and it can be obtained from  $s$  primitive recursively as well.

The normal form theorem shows that only a single unbounded search is required for the definition of any partial recursive function. Basically, we can search through all numbers until we find one that codes a computation of the function with index  $e$  for input  $x$ . We can use the numbers  $e$  as “names” of partial recursive functions, and write  $\varphi_e$  for the function  $f$  defined by the equation in the theorem. Note that any partial recursive function can have more than one index—in fact, every partial recursive function has infinitely many indices.

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## Bibliography