Theorem rec.1 (Kleene’s Normal Form Theorem). There is a primitive recursive relation $T(e, x, s)$ and a primitive recursive function $U(s)$, with the following property: if $f$ is any partial recursive function, then for some $e$,

$$f(x) \simeq U(\mu s T(e, x, s))$$

for every $x$.

The proof of the normal form theorem is involved, but the basic idea is simple. Every partial recursive function has an index $e$, intuitively, a number coding its program or definition. If $f(x) \downarrow$, the computation can be recorded systematically and coded by some number $s$, and that $s$ codes the computation of $f$ on input $x$ can be checked primitive recursively using only $x$ and the definition $e$. This means that $T$ is primitive recursive. Given the full record of the computation $s$, the “upshot” of $s$ is the value of $f(x)$, and it can be obtained from $s$ primitive recursively as well.

The normal form theorem shows that only a single unbounded search is required for the definition of any partial recursive function. We can use the numbers $e$ as “names” of partial recursive functions, and write $\phi_e$ for the function $f$ defined by the equation in the theorem. Note that any partial recursive function can have more than one index—in fact, every partial recursive function has infinitely many indices.

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Bibliography