

## rec.1 Non-Primitive Recursive Functions

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The primitive recursive functions do not exhaust the intuitively computable functions. It should be intuitively clear that we can make a list of all the unary primitive recursive functions,  $f_0, f_1, f_2, \dots$  such that we can effectively compute the value of  $f_x$  on input  $y$ ; in other words, the function  $g(x, y)$ , defined by

$$g(x, y) = f_x(y)$$

is computable. But then so is the function

$$\begin{aligned} h(x) &= g(x, x) + 1 \\ &= f_x(x) + 1. \end{aligned}$$

For each primitive recursive function  $f_i$ , the value of  $h$  and  $f_i$  differ at  $i$ . So  $h$  is computable, but not primitive recursive; and one can say the same about  $g$ . This is an “effective” version of Cantor’s diagonalization argument.

One can provide more explicit examples of computable functions that are not primitive recursive. For example, let the notation  $g^n(x)$  denote  $g(g(\dots g(x)))$ , with  $n$   $g$ ’s in all; and define a sequence  $g_0, g_1, \dots$  of functions by

$$\begin{aligned} g_0(x) &= x + 1 \\ g_{n+1}(x) &= g_n^x(x) \end{aligned}$$

You can confirm that each function  $g_n$  is primitive recursive. Each successive function grows much faster than the one before;  $g_1(x)$  is equal to  $2x$ ,  $g_2(x)$  is equal to  $2^x \cdot x$ , and  $g_3(x)$  grows roughly like an exponential stack of  $x$  2’s. Ackermann’s function is essentially the function  $G(x) = g_x(x)$ , and one can show that this grows faster than any primitive recursive function.

Let us return to the issue of enumerating the primitive recursive functions. Remember that we have assigned symbolic notations to each primitive recursive function; so it suffices to enumerate notations. We can assign a natural number  $\#(F)$  to each notation  $F$ , recursively, as follows:

$$\begin{aligned} \#(0) &= \langle 0 \rangle \\ \#(S) &= \langle 1 \rangle \\ \#(P_i^n) &= \langle 2, n, i \rangle \\ \#(\text{Comp}_{k,l}[H, G_0, \dots, G_{k-1}]) &= \langle 3, k, l, \#(H), \#(G_0), \dots, \#(G_{k-1}) \rangle \\ \#(\text{Rec}_l[G, H]) &= \langle 4, l, \#(G), \#(H) \rangle \end{aligned}$$

Here I am using the fact that every sequence of numbers can be viewed as a natural number, using the codes from the last section. The upshot is that every code is assigned a natural number. Of course, some sequences (and hence some numbers) do not correspond to notations; but we can let  $f_i$  be the unary primitive recursive function with notation coded as  $i$ , if  $i$  codes such a notation; and the constant 0 function otherwise. The net result is that we have an explicit way of enumerating the unary primitive recursive functions.

(In fact, some functions, like the constant zero function, will appear more than once on the list. This is not just an artifact of our coding, but also a result of the fact that the constant zero function has more than one notation. We will later see that one can not computably avoid these repetitions; for example, there is no computable function that decides whether or not a given notation represents the constant zero function.)

We can now take the function  $g(x, y)$  to be given by  $f_x(y)$ , where  $f_x$  refers to the enumeration we have just described. How do we know that  $g(x, y)$  is computable? Intuitively, this is clear: to compute  $g(x, y)$ , first “unpack”  $x$ , and see if it is a notation for a unary function; if it is, compute the value of that function on input  $y$ .

digression

You may already be convinced that (with some work!) one can write a program (say, in Java or C++) that does this; and now we can appeal to the Church-Turing thesis, which says that anything that, intuitively, is computable can be computed by a Turing machine.

Of course, a more direct way to show that  $g(x, y)$  is computable is to describe a Turing machine that computes it, explicitly. This would, in particular, avoid the Church-Turing thesis and appeals to intuition. But, as noted above, working with Turing machines directly is unpleasant. Soon we will have built up enough machinery to show that  $g(x, y)$  is computable, appealing to a model of computation that can be *simulated* on a Turing machine: namely, the recursive functions.

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## Bibliography