

rec.1 Bounded Minimization

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sec

It is often useful to define a function as the least number satisfying some property or relation P . If P is decidable, we can compute this function simply by trying out all the possible numbers, $0, 1, 2, \dots$, until we find the least one satisfying P . This kind of unbounded search takes us out of the realm of primitive recursive functions. However, if we're only interested in the least number *less than some independently given bound*, we stay primitive recursive. In other words, and a bit more generally, suppose we have a primitive recursive relation $R(x, z)$. Consider the function that maps y and z to the least $x < y$ such that $R(x, z)$. It, too, can be computed, by testing whether $R(0, z), R(1, z), \dots, R(y - 1, z)$. But why is it primitive recursive?

explanation

Proposition rec.1. *If $R(x, \vec{z})$ is primitive recursive, so is the function $m_R(y, \vec{z})$ which returns the least x less than y such that $R(x, \vec{z})$ holds, if there is one, and 0 otherwise. We will write the function m_R as*

$$(\min x < y) R(x, \vec{z}),$$

Proof. Note that there can be no $x < 0$ such that $R(x, \vec{z})$ since there is no $x < 0$ at all. So $m_R(x, 0) = 0$.

In case the bound is $y + 1$ we have three cases: (a) There is an $x < y$ such that $R(x, \vec{z})$, in which case $m_R(y + 1, \vec{z}) = m_R(y, \vec{z})$. (b) There is no such x but $R(y, \vec{z})$ holds, then $m_R(y + 1, \vec{z}) = y$. (c) There is no $x < y + 1$ such that $R(x, \vec{z})$, then $m_R(y + 1, \vec{z}) = 0$. So,

$$m_R(0, \vec{z}) = 0$$
$$m_R(y + 1, \vec{z}) = \begin{cases} m_R(y, \vec{z}) & \text{if } (\exists x < y) R(x, \vec{z}) \\ y & \text{otherwise, provided } R(y, \vec{z}) \\ 0 & \text{otherwise.} \end{cases}$$

□

The choice of “0 otherwise” is somewhat arbitrary. It is in fact even easier to recursively define the function m'_R which returns the least x less than y such that $R(x, \vec{z})$ holds, and $y + 1$ otherwise. When we use \min , however, we will always know that the least x such that $R(x, \vec{z})$ exists and is less than y . Thus, in practice, we will not have to worry about the possibility that if $(\min x < y) R(x, \vec{z}) = 0$ we do not know if that value indicates that $R(0, \vec{z})$ or that for no $x < y$, $R(x, \vec{z})$. As with bounded quantification, $(\min x \leq y) \dots$ can be understood as $(\min x < y + 1) \dots$

explanation

Problem rec.1. Suppose $R(x, \vec{z})$ is primitive recursive. Define the function $m'_R(y, \vec{z})$ which returns the least x less than y such that $R(x, \vec{z})$ holds, if there is one, and $y + 1$ otherwise, by primitive recursion from χ_R .

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Bibliography