rec.1 Bounded Minimization

It is often useful to define a function as the least number satisfying some property or relation \( P \). If \( P \) is decidable, we can compute this function simply by trying out all the possible numbers, 0, 1, 2, \ldots, until we find the least one satisfying \( P \). This kind of unbounded search takes us out of the realm of primitive recursive functions. However, if we’re only interested in the least number \textit{less than some independently given bound}, we stay primitive recursive. In other words, and a bit more generally, suppose we have a primitive recursive relation \( R(x, z) \). Consider the function that maps \( x \) and \( y \) to the least \( z < y \) such that \( R(x, z) \). It, too, can be computed, by testing whether \( R(x, 0), R(x, 1), \ldots, R(x, y - 1) \). But why is it primitive recursive?

**Proposition rec.1.** If \( R(\vec{x}, z) \) is primitive recursive, so is the function \( m_R(\vec{x}, y) \) which returns the least \( z < y \) such that \( R(\vec{x}, z) \) holds, if there is one, and \( y \) otherwise. We will write the function \( m_R \) as

\[
(\min z < y) R(\vec{x}, z),
\]

**Proof.** Note than there can be no \( z < 0 \) such that \( R(\vec{x}, z) \) since there is no \( z < 0 \) at all. So \( m_R(\vec{x}, 0) = 0 \).

In case the bound is of the form \( y + 1 \) we have three cases:

1. There is a \( z < y \) such that \( R(\vec{x}, z) \), in which case \( m_R(\vec{x}, y+1) = m_R(\vec{x}, y) \).
2. There is no such \( z < y \) but \( R(\vec{x}, y) \) holds, then \( m_R(\vec{x}, y + 1) = y \).
3. There is no \( z < y + 1 \) such that \( R(\vec{x}, z) \), then \( m_R(\vec{x}, y+1) = y + 1 \).

So we can define \( m_R(\vec{x}, 0) \) by primitive recursion as follows:

\[
m_R(\vec{x}, 0) = 0
\]

\[
m_R(\vec{x}, y + 1) = \begin{cases} 
m_R(\vec{x}, y) & \text{if } m_R(\vec{x}, y) \neq y \\
y & \text{if } m_R(\vec{x}, y) = y \text{ and } R(\vec{x}, y) \\
y + 1 & \text{otherwise.}
\end{cases}
\]

Note that there is a \( z < y \) such that \( R(\vec{x}, z) \) iff \( m_R(\vec{x}, y) \neq y \).

**Problem rec.1.** Suppose \( R(\vec{x}, z) \) is primitive recursive. Define the function \( m'_R(\vec{x}, y) \) which returns the least \( z \) less than \( y \) such that \( R(\vec{x}, z) \) holds, if there is one, and 0 otherwise, by primitive recursion from \( \chi_R \).

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Bibliography