

## lam.1 Reduction of Lambda Terms

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sec

What can one do with lambda terms? Simplify them. If  $M$  and  $N$  are any lambda terms and  $x$  is any variable, we can use  $M[N/x]$  to denote the result of substituting  $N$  for  $x$  in  $M$ , after renaming any bound variables of  $M$  that would interfere with the free variables of  $N$  after the substitution. For example,

$$(\lambda w. xxw)[yyz/x] = \lambda w. (yyz)(yyz)w.$$

Alternative notations for substitution are  $[N/x]M$ ,  $M[N/x]$ , and also  $M[x/N]$ <sup>digression</sup>  
Beware!

Intuitively,  $(\lambda x. M)N$  and  $M[N/x]$  have the same meaning; the act of replacing the first term by the second is called  $\beta$ -conversion. More generally, if it is possible to convert a term  $P$  to  $P'$  by  $\beta$ -conversion of some subterm, one says  $P$   $\beta$ -reduces to  $P'$  in one step. If  $P$  can be converted to  $P'$  with any number of one-step reductions (possibly none), then  $P$   $\beta$ -reduces to  $P'$ . A term that cannot be  $\beta$ -reduced any further is called  $\beta$ -irreducible, or  $\beta$ -normal. I will say “reduces” instead of “ $\beta$ -reduces,” etc., when the context is clear.

Let us consider some examples.

1. We have

$$\begin{aligned} (\lambda x. xxy)\lambda z. z &\triangleright_1 (\lambda z. z)(\lambda z. z)y \\ &\triangleright_1 (\lambda z. z)y \\ &\triangleright_1 y \end{aligned}$$

2. “Simplifying” a term can make it more complex:

$$\begin{aligned} (\lambda x. xxy)(\lambda x. xxy) &\triangleright_1 (\lambda x. xxy)(\lambda x. xxy)y \\ &\triangleright_1 (\lambda x. xxy)(\lambda x. xxy)yy \\ &\triangleright_1 \dots \end{aligned}$$

3. It can also leave a term unchanged:

$$(\lambda x. xx)(\lambda x. xx) \triangleright_1 (\lambda x. xx)(\lambda x. xx)$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x. (\lambda y. yx)z)v \triangleright_1 (\lambda y. yv)z$$

by contracting the outermost application; and

$$(\lambda x. (\lambda y. yx)z)v \triangleright_1 (\lambda x. zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term,  $zv$ .

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the “Church-Rosser property.”

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## **Bibliography**