Let us consider one more example of using the s-m-n theorem to show that something is noncomputable. Let Tot be the set of indices of total computable functions, i.e.

$$\text{Tot} = \{x : \text{for every } y, \varphi_x(y) \downarrow\}.$$ 

**Proposition thy.1.** Tot is not computable.

**Proof.** To see that Tot is not computable, it suffices to show that $K$ is reducible to it. Let $h(x, y)$ be defined by

$$h(x, y) \simeq \begin{cases} 
0 & \text{if } x \in K \\
\text{undefined} & \text{otherwise} 
\end{cases}$$

Note that $h(x, y)$ does not depend on $y$ at all. It should not be hard to see that $h$ is partial computable: on input $x, y$, the we compute $h$ by first simulating the function $\varphi_x$ on input $x$; if this computation halts, $h(x, y)$ outputs 0 and halts. So $h(x, y)$ is just $Z(\mu s T(x, x, s))$, where $Z$ is the constant zero function.

Using the s-m-n theorem, there is a primitive recursive function $k(x)$ such that for every $x$ and $y$,

$$\varphi_{k(x)}(y) \simeq \begin{cases} 
0 & \text{if } x \in K \\
\text{undefined} & \text{otherwise} 
\end{cases}$$

So $\varphi_{k(x)}$ is total if $x \in K$, and undefined otherwise. Thus, $k$ is a reduction of $K$ to Tot.

It turns out that Tot is not even computably enumerable—its complexity lies further up on the “arithmetic hierarchy.” But we will not worry about this strengthening here.

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**Bibliography**