

## thy.1 Totality is Undecidable

cmp:thy:tot:  
sec

Let us consider one more example of using the  $s$ - $m$ - $n$  theorem to show that something is noncomputable. Let Tot be the set of indices of total computable functions, i.e.

$$\text{Tot} = \{x : \text{for every } y, \varphi_x(y) \downarrow\}.$$

cmp:thy:tot:  
prop:total

**Proposition thy.1.** *Tot is not computable.*

*Proof.* To see that Tot is not computable, it suffices to show that  $K$  is reducible to it. Let  $h(x, y)$  be defined by

$$h(x, y) \simeq \begin{cases} 0 & \text{if } x \in K \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that  $h(x, y)$  does not depend on  $y$  at all. It should not be hard to see that  $h$  is partial computable: on input  $x, y$ , we compute  $h$  by first simulating the function  $\varphi_x$  on input  $x$ ; if this computation halts,  $h(x, y)$  outputs 0 and halts. So  $h(x, y)$  is just  $Z(\mu s T(x, x, s))$ , where  $Z$  is the constant zero function.

Using the  $s$ - $m$ - $n$  theorem, there is a primitive recursive function  $k(x)$  such that for every  $x$  and  $y$ ,

$$\varphi_{k(x)}(y) = \begin{cases} 0 & \text{if } x \in K \\ \text{undefined} & \text{otherwise} \end{cases}$$

So  $\varphi_{k(x)}$  is total if  $x \in K$ , and undefined otherwise. Thus,  $k$  is a reduction of  $K$  to Tot.  $\square$

It turns out that Tot is not even computably enumerable—its complexity [digression](#) lies further up on the “arithmetic hierarchy.” But we will not worry about this strengthening here.

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## Bibliography