Theorem thy.1  The s-m-n Theorem

The next theorem is known as the “s-m-n theorem,” for a reason that will be clear in a moment. The hard part is understanding just what the theorem says; once you understand the statement, it will seem fairly obvious.

Theorem thy.1. For each pair of natural numbers \( n \) and \( m \), there is a primitive recursive function \( s^m_n \) such that for every sequence \( x, a_0, \ldots, a_{m-1}, y_0, \ldots, y_{n-1} \), we have

\[
\phi_n^{s^m_n(x,a_0,\ldots,a_{m-1})}(y_0,\ldots,y_{n-1}) \simeq \phi_x^{m+n}(a_0,\ldots,a_{m-1},y_0,\ldots,y_{n-1}).
\]

It is helpful to think of \( s^m_n \) as acting on programs. That is, \( s^m_n \) takes a program, \( x \), for an \((m + n)\)-ary function, as well as fixed inputs \( a_0, \ldots, a_{m-1} \); and it returns a program, \( s^m_n(x,a_0,\ldots,a_{m-1}) \), for the \( n \)-ary function of the remaining arguments. If you think of \( x \) as the description of a Turing machine, then \( s^m_n(x,a_0,\ldots,a_{m-1}) \) is the Turing machine that, on input \( y_0, \ldots, y_{n-1} \), prepends \( a_0, \ldots, a_{m-1} \) to the input string, and runs \( x \). Each \( s^m_n \) is then just a primitive recursive function that finds a code for the appropriate Turing machine.

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Bibliography