

thy.1 Comparison with Russell's Paradox

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sec It is instructive to compare and contrast the arguments in this section with Russell's paradox:

1. Russell's paradox: let $S = \{x : x \notin x\}$. Then $S \in S$ if and only if $S \notin S$, a contradiction.

Conclusion: There is no such set S . Assuming the existence of a “set of all sets” is inconsistent with the other axioms of set theory.

2. A modification of Russell's paradox: let F be the “function” from the set of all functions to $\{0, 1\}$, defined by

$$F(f) = \begin{cases} 1 & \text{if } f \text{ is in the domain of } f, \text{ and } f(f) = 0 \\ 0 & \text{otherwise} \end{cases}$$

A similar argument shows that $F(F) = 0$ if and only if $F(F) = 1$, a contradiction.

Conclusion: F is not a function. The “set of all functions” is too big to be the domain of a function.

3. The diagonalization argument: let f_0, f_1, \dots be the enumeration of the partial computable functions, and let $G: \mathbb{N} \rightarrow \{0, 1\}$ be defined by

$$G(x) = \begin{cases} 1 & \text{if } f_x(x) \downarrow = 0 \\ 0 & \text{otherwise} \end{cases}$$

If G is computable, then it is the function f_k for some k . But then $G(k) = 1$ if and only if $G(k) = 0$, a contradiction.

Conclusion: G is not computable. Note that according to the axioms of set theory, G is still a function; there is no paradox here, just a clarification.

That talk of partial functions, computable functions, partial computable functions, and so on can be confusing. The set of all partial functions from \mathbb{N} to \mathbb{N} is a big collection of objects. Some of them are total, some of them are computable, some are both total and computable, and some are neither. Keep in mind that when we say “function,” by default, we mean a total function. Thus we have:

1. computable functions
2. partial computable functions that are not total
3. functions that are not computable
4. partial functions that are neither total nor computable

To sort this out, it might help to draw a big square representing all the partial functions from \mathbb{N} to \mathbb{N} , and then mark off two overlapping regions, corresponding to the total functions and the computable partial functions, respectively. It is a good exercise to see if you can describe an object in each of the resulting regions in the diagram.

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Bibliography