

## thy.1 Reducibility

cmp:thy:red:sec We now know that there is at least one set,  $K_0$ , that is computably enumerable explanation but not computable. It should be clear that there are others. The method of reducibility provides a powerful method of showing that other sets have these properties, without constantly having to return to first principles.

Generally speaking, a “reduction” of a set  $A$  to a set  $B$  is a method of transforming answers to whether or not elements are in  $B$  into answers as to whether or not elements are in  $A$ . We will focus on a notion called “many-one reducibility,” but there are many other notions of reducibility available, with varying properties. Notions of reducibility are also central to the study of computational complexity, where efficiency issues have to be considered as well. For example, a set is said to be “NP-complete” if it is in NP and every NP problem can be reduced to it, using a notion of reduction that is similar to the one described below, only with the added requirement that the reduction can be computed in polynomial time.

We have already used this notion implicitly. Define the set  $K$  by

$$K = \{x : \varphi_x(x) \downarrow\},$$

i.e.,  $K = \{x : x \in W_x\}$ . Our proof that the halting problem is unsolvable, ??, shows most directly that  $K$  is not computable. Recall that  $K_0$  is the set

$$K_0 = \{\langle e, x \rangle : \varphi_e(x) \downarrow\}.$$

i.e.  $K_0 = \{\langle x, e \rangle : x \in W_e\}$ . It is easy to extend any proof of the uncomputability of  $K$  to the uncomputability of  $K_0$ : if  $K_0$  were computable, we could decide whether or not an element  $x$  is in  $K$  simply by asking whether or not the pair  $\langle x, x \rangle$  is in  $K_0$ . The function  $f$  which maps  $x$  to  $\langle x, x \rangle$  is an example of a *reduction* of  $K$  to  $K_0$ .

**Definition thy.1.** Let  $A$  and  $B$  be sets. Then  $A$  is said to be *many-one reducible* to  $B$ , written  $A \leq_m B$ , if there is a computable function  $f$  such that for every natural number  $x$ ,

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

If  $A$  is many-one reducible to  $B$  and vice-versa, then  $A$  and  $B$  are said to be *many-one equivalent*, written  $A \equiv_m B$ .

If the function  $f$  in the definition above happens to be injective,  $A$  is said to be *one-one reducible* to  $B$ . Most of the reductions described below meet this stronger requirement, but we will not use this fact.

It is true, but by no means obvious, that one-one reducibility really is a digression stronger requirement than many-one reducibility. In other words, there are infinite sets  $A$  and  $B$  such that  $A$  is many-one reducible to  $B$  but not one-one reducible to  $B$ .

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**Bibliography**