Theorem thy.1 (Kleene’s Normal Form Theorem). There are a primitive recursive relation \( T(k, x, s) \) and a primitive recursive function \( U(s) \), with the following property: if \( f \) is any partial computable function, then for some \( k \),

\[
f(x) \simeq U(\mu s \ T(k, x, s))
\]

for every \( x \).

Proof Sketch. For any model of computation one can rigorously define a description of the computable function \( f \) and code such description using a natural number \( k \). One can also rigorously define a notion of “computation sequence” which records the process of computing the function with index \( k \) for input \( x \). These computation sequences can likewise be coded as numbers \( s \). This can be done in such a way that (a) it is decidable whether a number \( s \) codes the computation sequence of the function with index \( k \) on input \( x \) and (b) what the end result of the computation sequence coded by \( s \) is. In fact, the relation in (a) and the function in (b) are primitive recursive.

In order to give a rigorous proof of the Normal Form Theorem, we would have to fix a model of computation and carry out the coding of descriptions of computable functions and of computation sequences in detail, and verify that the relation \( T \) and function \( U \) are primitive recursive. For most applications, it suffices that \( T \) and \( U \) are computable and that \( U \) is total.

It is probably best to remember the proof of the normal form theorem in slogan form: \( \mu s \ T(k, x, s) \) searches for a computation sequence of the function with index \( k \) on input \( x \), and \( U \) returns the output of the computation sequence if one can be found.

\( T \) and \( U \) can be used to define the enumeration \( \varphi_0, \varphi_1, \varphi_2, \ldots \). From now on, we will assume that we have fixed a suitable choice of \( T \) and \( U \), and take the equation

\[
\varphi_e(x) \simeq U(\mu s \ T(e, x, s))
\]

to be the definition of \( \varphi_e \).

Here is another useful fact:

Theorem thy.2. Every partial computable function has infinitely many indices.

Again, this is intuitively clear. Given any (description of) a computable function, one can come up with a different description which computes the same function (input-output pair) but does so, e.g., by first doing something that has no effect on the computation (say, test if \( 0 = 0 \), or count to 5, etc.). The index of the altered description will always be different from the original index. Both are indices of the same function, just computed slightly differently.
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Bibliography