thy.1 There Are Non-Computable Sets

cmp:thy:ncp:sec We saw above that every computable set is computably enumerable. Is the converse true? The following shows that, in general, it is not.

cmp:thy:ncp:thm:K-0 **Theorem thy.1.** *Let K_0 be the set $\{\langle e, x\rangle : \varphi_e(x)\downarrow\}$. Then K_0 is computably enumerable but not computable.*

Proof. To see that K_0 is computably enumerable, note that it is the domain of the function f defined by

$$f(z) = \mu y\;(\mathrm{len}(z) = 2 \wedge T((z)_0, (z)_1, y)).$$

For, if $\varphi_e(x)$ is defined, $f(\langle e, x\rangle)$ finds a halting computation sequence; if $\varphi_e(x)$ is undefined, so is $f(\langle e, x\rangle)$; and if z doesn't even code a pair, then $f(z)$ is also undefined.

The fact that K_0 is not computable is just the undecidability of the halting problem, **??**. □

The set K_0 is the set of pairs $\langle e, x\rangle$ such that $\varphi_e(x)\downarrow$, i.e., $\langle e, x\rangle \in K_0$ iff φ_e is defined (halts) on input x, so it is also called the "halting set." The set $K = \{e : \varphi_e(e)\downarrow\}$ is the "self-halting set." It is often used as a canonical undecidable set.

cmp:thy:ncp:thm:K **Theorem thy.2.** *The self-halting set $K = \{e : \varphi_e(e)\downarrow\}$ is c.e. but not decidable.*

Proof. Suppose K is decidable, i.e., its characteristic function χ_K is computable. Let

$$d(e) = \begin{cases} 1 & \text{if } \chi_K(e) = 0 \\ \uparrow & \text{otherwise.} \end{cases}$$

Let k be the index of d, i.e., $d \simeq \varphi_k$. Then $d(k) \simeq \varphi_k(k)$. This contradicts the fact that $d(k)\downarrow$ iff $\varphi_k(k)\uparrow$, which follows from the definition of d.

K is the domain of $f(x) = \mu y\; T(x, x, y)$ and so is c.e. □

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Bibliography