When it comes to the lambda calculus, we’ve shown the following:

1. Every primitive recursive function is represented by a lambda term.
2. There is a lambda term $Y$ such that for any lambda term $G$, $YG \rightarrow G(YG)$.

To show that every partial computable function is represented by some lambda term, we only need to show the following.

**Lemma thy.1.** Suppose $f(x, y)$ is primitive recursive. Let $g$ be defined by $g(x) \simeq \mu y f(x, y) = 0$.

Then $g$ is represented by a lambda term.

**Proof.** The idea is roughly as follows. Given $x$, we will use the fixed-point lambda term $Y$ to define a function $h_x(n)$ which searches for a $y$ starting at $n$; then $g(x)$ is just $h_x(0)$. The function $h_x$ can be expressed as the solution of a fixed-point equation:

$$h_x(n) \simeq \begin{cases} n & \text{if } f(x, n) = 0 \\ h_x(n + 1) & \text{otherwise.} \end{cases}$$

Here are the details. Since $f$ is primitive recursive, it is represented by some term $F$. Remember that we also have a lambda term $D$ such that $D(M, N, 0) \rightarrow M$ and $D(M, N, 1) \rightarrow N$. Fixing $x$ for the moment, to represent $h_x$ we want to find a term $H$ (depending on $x$) satisfying

$$H(n) \equiv D(n, H(S(n)), F(x, n)).$$

We can do this using the fixed-point term $Y$. First, let $U$ be the term

$$\lambda h. \lambda z. D(z, (h(Sz)), F(x, z)),$$

and then let $H$ be the term $YU$. Notice that the only free variable in $H$ is $x$. Let us show that $H$ satisfies the equation above.

By the definition of $Y$, we have

$$H = YU \equiv U(YU) = U(H).$$

In particular, for each natural number $n$, we have

$$H(n) \equiv U(H, n) \rightarrow D(n, H(S(n)), F(x, n)),$$

as required. Notice that if you substitute a numeral $\overline{m}$ for $x$ in the last line, the expression reduces to $\overline{n}$ if $F(\overline{m}, \overline{n})$ reduces to $\overline{0}$, and it reduces to $H(S(\overline{n}))$ if $F(\overline{m}, \overline{n})$ reduces to any other numeral.
To finish off the proof, let $G$ be $\lambda x. H(\overline{0})$. Then $G$ represents $g$; in other words, for every $m$, $G(m)$ reduces to $g(m)$, if $g(m)$ is defined, and has no normal form otherwise.

\[\square\]

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Bibliography