

## thy.1 Minimization with Lambda Terms

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When it comes to the lambda calculus, we've shown the following:

1. Every primitive recursive function is represented by a lambda term.
2. There is a lambda term  $Y$  such that for any lambda term  $G$ ,  $YG \rightarrow G(YG)$ .

To show that every partial computable function is represented by some lambda term, we only need to show the following.

**Lemma thy.1.** *Suppose  $f(x, y)$  is primitive recursive. Let  $g$  be defined by*

$$g(x) \simeq \mu y f(x, y) = 0.$$

*Then  $g$  is represented by a lambda term.*

*Proof.* The idea is roughly as follows. Given  $x$ , we will use the fixed-point lambda term  $Y$  to define a function  $h_x(n)$  which searches for a  $y$  starting at  $n$ ; then  $g(x)$  is just  $h_x(0)$ . The function  $h_x$  can be expressed as the solution of a fixed-point equation:

$$h_x(n) \simeq \begin{cases} n & \text{if } f(x, n) = 0 \\ h_x(n+1) & \text{otherwise.} \end{cases}$$

Here are the details. Since  $f$  is primitive recursive, it is represented by some term  $F$ . Remember that we also have a lambda term  $D$  such that  $D(M, N, \bar{0}) \rightarrow M$  and  $D(M, N, \bar{1}) \rightarrow N$ . Fixing  $x$  for the moment, to represent  $h_x$  we want to find a term  $H$  (depending on  $x$ ) satisfying

$$H(\bar{n}) \equiv D(\bar{n}, H(S(\bar{n})), F(x, \bar{n})).$$

We can do this using the fixed-point term  $Y$ . First, let  $U$  be the term

$$\lambda h. \lambda z. D(z, (h(Sz)), F(x, z)),$$

and then let  $H$  be the term  $YU$ . Notice that the only free variable in  $H$  is  $x$ . Let us show that  $H$  satisfies the equation above.

By the definition of  $Y$ , we have

$$H = YU \equiv U(YU) = U(H).$$

In particular, for each natural number  $n$ , we have

$$\begin{aligned} H(\bar{n}) &\equiv U(H, \bar{n}) \\ &\rightarrow D(\bar{n}, H(S(\bar{n})), F(x, \bar{n})), \end{aligned}$$

as required. Notice that if you substitute a numeral  $\bar{m}$  for  $x$  in the last line, the expression reduces to  $\bar{n}$  if  $F(\bar{m}, \bar{n})$  reduces to  $\bar{0}$ , and it reduces to  $H(S(\bar{n}))$  if  $F(\bar{m}, \bar{n})$  reduces to any other numeral.

To finish off the proof, let  $G$  be  $\lambda x. H(\bar{0})$ . Then  $G$  represents  $g$ ; in other words, for every  $m$ ,  $G(\bar{m})$  reduces to  $\bar{g(m)}$ , if  $g(m)$  is defined, and has no normal form otherwise.  $\square$

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## Bibliography