thry.1 Minimization with Lambda Terms

When it comes to the lambda calculus, we’ve shown the following:

1. Every primitive recursive function is represented by a lambda term.

2. There is a lambda term \( Y \) such that for any lambda term \( G \), \( YG \to G(YG) \).

To show that every partial computable function is represented by some lambda term, we only need to show the following.

Lemma thy.1. Suppose \( f(x, y) \) is primitive recursive. Let \( g \) be defined by

\[
g(x) \simeq \mu y \, f(x, y) = 0.
\]

Then \( g \) is represented by a lambda term.

Proof. The idea is roughly as follows. Given \( x \), we will use the fixed-point lambda term \( Y \) to define a function \( h_x(n) \) which searches for a \( y \) starting at \( n \); then \( g(x) \) is just \( h_x(0) \). The function \( h_x \) can be expressed as the solution of a fixed-point equation:

\[
h_x(n) \simeq \begin{cases} n & \text{if } f(x, n) = 0 \\ h_x(n + 1) & \text{otherwise.} \end{cases}
\]

Here are the details. Since \( f \) is primitive recursive, it is represented by some term \( F \). Remember that we also have a lambda term \( D \) such that \( D(M, N, 0) \to M \) and \( D(M, N, 1) \to N \). Fixing \( x \) for the moment, to represent \( h_x \) we want to find a term \( H \) (depending on \( x \)) satisfying

\[
H(\overline{n}) \equiv D(\overline{n}, H(S(\overline{n})), F(x, \overline{n})).
\]

We can do this using the fixed-point term \( Y \). First, let \( U \) be the term

\[
\lambda h. \lambda z. D(z, (h(Sz)), F(x, z)),
\]

and then let \( H \) be the term \( YU \). Notice that the only free variable in \( H \) is \( x \). Let us show that \( H \) satisfies the equation above.

By the definition of \( Y \), we have

\[
H = YU \equiv U(YU) = U(H).
\]

In particular, for each natural number \( n \), we have

\[
H(\overline{n}) \equiv U(H, \overline{n}) \\
\to D(\overline{n}, H(S(\overline{n})), F(x, \overline{n})),
\]

as required. Notice that if you substitute a numeral \( \overline{m} \) for \( x \) in the last line, the expression reduces to \( \overline{n} \) if \( F(\overline{m}, \overline{n}) \) reduces to \( 0 \), and it reduces to \( H(S(\overline{n})) \) if \( F(\overline{m}, \overline{n}) \) reduces to any other numeral.
To finish off the proof, let $G$ be $\lambda x. H(\overline{u})$. Then $G$ represents $g$; in other words, for every $m$, $G(m)$ reduces to $g(m)$, if $g(m)$ is defined, and has no normal form otherwise.

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Bibliography