

## thy.1 An Example of Reducibility

cmp:thy:k1: Let us consider an application of ??.  
sec

cmp:thy:k1: **Proposition thy.1.** *Let*  
prop:k1

$$K_1 = \{e : \varphi_e(0) \downarrow\}.$$

*Then  $K_1$  is computably enumerable but not computable.*

*Proof.* Since  $K_1 = \{e : \exists s T(e, 0, s)\}$ ,  $K_1$  is computably enumerable by ??.

To show that  $K_1$  is not computable, let us show that  $K_0$  is reducible to it.

This is a little bit tricky, since using  $K_1$  we can only ask questions about computations that start with a particular input, 0. Suppose you have a smart friend who can answer questions of this type (friends like this are known as “oracles”). Then suppose someone comes up to you and asks you whether or not  $\langle e, x \rangle$  is in  $K_0$ , that is, whether or not machine  $e$  halts on input  $x$ . One thing you can do is build another machine,  $e_x$ , that, for *any* input, ignores that input and instead runs  $e$  on input  $x$ . Then clearly the question as to whether machine  $e$  halts on input  $x$  is equivalent to the question as to whether machine  $e_x$  halts on input 0 (or any other input). So, then you ask your friend whether this new machine,  $e_x$ , halts on input 0; your friend’s answer to the modified question provides the answer to the original one. This provides the desired reduction of  $K_0$  to  $K_1$ . explanation

Using the universal partial computable function, let  $f$  be the 3-ary function defined by

$$f(x, y, z) \simeq \varphi_x(y).$$

Note that  $f$  ignores its third input entirely. Pick an index  $e$  such that  $f = \varphi_e^3$ ; so we have

$$\varphi_e^3(x, y, z) \simeq \varphi_x(y).$$

By the  $s$ - $m$ - $n$  theorem, there is a function  $s(e, x, y)$  such that, for every  $z$ ,

$$\begin{aligned} \varphi_{s(e,x,y)}(z) &\simeq \varphi_e^3(x, y, z) \\ &\simeq \varphi_x(y). \end{aligned}$$

In terms of the informal argument above,  $s(e, x, y)$  is an index for the machine that, for any input  $z$ , ignores that input and computes  $\varphi_x(y)$ . explanation

In particular, we have

$$\varphi_{s(e,x,y)}(0) \downarrow \quad \text{if and only if} \quad \varphi_x(y) \downarrow.$$

In other words,  $\langle x, y \rangle \in K_0$  if and only if  $s(e, x, y) \in K_1$ . So the function  $g$  defined by

$$g(w) = s(e, (w)_0, (w)_1)$$

is a reduction of  $K_0$  to  $K_1$ . □

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**Bibliography**