An Example of Reducibility

Let us consider an application of ??.

**Proposition thy.1.** Let

\[ K_1 = \{ e : \varphi_e(0) \downarrow \}. \]

Then \( K_1 \) is computably enumerable but not computable.

**Proof.** Since \( K_1 = \{ e : \exists s T(e, 0, s) \} \), \( K_1 \) is computably enumerable by ??.

To show that \( K_1 \) is not computable, let us show that \( K_0 \) is reducible to it.

This is a little bit tricky, since using \( K_1 \) we can only ask questions about computations that start with a particular input, 0. Suppose you have a smart friend who can answer questions of this type (friends like this are known as “oracles”). Then suppose someone comes up to you and asks you whether or not \( \langle e, x \rangle \) is in \( K_0 \), that is, whether or not machine \( e \) halts on input \( x \). One thing you can do is build another machine, \( e_x \), that, for *any* input, ignores that input and instead runs \( e \) on input \( x \). Then clearly the question as to whether machine \( e \) halts on input \( x \) is equivalent to the question as to whether machine \( e_x \) halts on input 0 (or any other input). So, then you ask your friend whether this new machine, \( e_x \), halts on input 0; your friend’s answer to the modified question provides the answer to the original one. This provides the desired reduction of \( K_0 \) to \( K_1 \).

Using the universal partial computable function, let \( f \) be the 3-ary function defined by

\[ f(x, y, z) \simeq \varphi_x(y). \]

Note that \( f \) ignores its third input entirely. Pick an index \( e \) such that \( f = \varphi^3_e \); so we have

\[ \varphi^3_e(x, y, z) \simeq \varphi_x(y). \]

By the \( s-m-n \) theorem, there is a function \( s(e, x, y) \) such that, for every \( z \),

\[ \varphi_{s(e,x,y)}(z) \simeq \varphi^3_e(x, y, z) \simeq \varphi_x(y). \]

In terms of the informal argument above, \( s(e, x, y) \) is an index for the machine that, for any input \( z \), ignores that input and computes \( \varphi_x(y) \).

In particular, we have

\[ \varphi_{s(e,x,y)}(0) \downarrow \text{ if and only if } \varphi_x(y) \downarrow. \]

In other words, \( \langle x, y \rangle \in K_0 \) if and only if \( s(e, x, y) \in K_1 \). So the function \( g \) defined by

\[ g(w) = s(e, (w)_0, (w)_1) \]

is a reduction of \( K_0 \) to \( K_1 \). \qed