

thy.1 An Example of Reducibility

Let us consider an application of ??.

Proposition thy.1. *Let*

$$K_1 = \{e : \varphi_e(0) \downarrow\}.$$

Then K_1 is computably enumerable but not computable.

Proof. Since $K_1 = \{e : \exists s T(e, 0, s)\}$, K_1 is computably enumerable by ??.

To show that K_1 is not computable, let us show that K_0 is reducible to it.

This is a little bit tricky, since using K_1 we can only ask questions about computations that start with a particular input, 0. Suppose you have a smart friend who can answer questions of this type (friends like this are known as “oracles”). Then suppose someone comes up to you and asks you whether or not $\langle e, x \rangle$ is in K_0 , that is, whether or not machine e halts on input x . One thing you can do is build another machine, e_x , that, for *any* input, ignores that input and instead runs e on input x . Then clearly the question as to whether machine e halts on input x is equivalent to the question as to whether machine e_x halts on input 0 (or any other input). So, then you ask your friend whether this new machine, e_x , halts on input 0; your friend’s answer to the modified question provides the answer to the original one. This provides the desired reduction of K_0 to K_1 .

Using the universal partial computable function, let f be the 3-ary function defined by

$$f(x, y, z) \simeq \varphi_x(y).$$

Note that f ignores its third input entirely. Pick an index e such that $f = \varphi_e^3$; so we have

$$\varphi_e^3(x, y, z) \simeq \varphi_x(y).$$

By the s - m - n theorem, there is a function $s(e, x, y)$ such that, for every z ,

$$\begin{aligned} \varphi_{s(e,x,y)}(z) &\simeq \varphi_e^3(x, y, z) \\ &\simeq \varphi_x(y). \end{aligned}$$

In terms of the informal argument above, $s(e, x, y)$ is an index for the machine that, for any input z , ignores that input and computes $\varphi_x(y)$.

In particular, we have

$$\varphi_{s(e,x,y)}(0) \downarrow \quad \text{if and only if} \quad \varphi_x(y) \downarrow.$$

In other words, $\langle x, y \rangle \in K_0$ if and only if $s(e, x, y) \in K_1$. So the function g defined by

$$g(w) = s(e, (w)_0, (w)_1)$$

is a reduction of K_0 to K_1 . □

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Bibliography