Let’s consider the halting problem again. As temporary notation, let us write \( \langle x, y \rangle \) for \( \phi_x(y) \); think of this as representing a “name” for the value \( \phi_x(y) \).

With this notation, we can reword one of our proofs that the halting problem is undecidable.

**Question:** is there a computable function \( h \), with the following property? For every \( x \) and \( y \),

\[
h(\langle \varphi_x(y) \rangle) = \begin{cases} 
1 & \text{if } \varphi_x(y) \downarrow \\
0 & \text{otherwise.}
\end{cases}
\]

**Answer:** No; otherwise, the partial function

\[
g(x) = \begin{cases} 
0 & \text{if } h(\langle \varphi_x(x) \rangle) = 0 \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

would be computable, and so have some index \( e \). But then we have

\[
\varphi_e(e) = \begin{cases} 
0 & \text{if } h(\langle \varphi_e(e) \rangle) = 0 \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

in which case \( \varphi_e(e) \) is defined if and only if it isn’t, a contradiction.

Now, take a look at the equation with \( \varphi_e \). There is an instance of self-reference there, in a sense: we have arranged for the value of \( \varphi_e(e) \) to depend on \( \langle \varphi_e(e) \rangle \), in a certain way. The fixed-point theorem says that we can do this, in general—not just for the sake of proving contradictions.

**Lemma thy.1** gives two equivalent ways of stating the fixed-point theorem.

Logically speaking, the fact that the statements are equivalent follows from the fact that they are both true; but what we really mean is that each one follows straightforwardly from the other, so that they can be taken as alternative statements of the same theorem.

**Lemma thy.1.** The following statements are equivalent:

1. For every partial computable function \( g(x, y) \), there is an index \( e \) such that for every \( y \),

\[
\varphi_e(y) \simeq g(e, y).
\]

2. For every computable function \( f(x) \), there is an index \( e \) such that for every \( y \),

\[
\varphi_e(y) \simeq \varphi_{f(e)}(y).
\]

**Proof.** \( (1) \Rightarrow (2) \): Given \( f \), define \( g \) by \( g(x, y) \simeq \text{Un}(f(x), y) \). Use \( (1) \) to get an index \( e \) such that for every \( y \),

\[
\varphi_e(y) = \text{Un}(f(e), y) = \varphi_{f(e)}(y).
\]
(2) ⇒ (1): Given \( g \), use the \( s\)-\( m\)-\( n \) theorem to get \( f \) such that for every \( x \) and \( y \), \( \varphi_{f(x)}(y) \simeq g(x, y) \). Use (2) to get an index \( e \) such that
\[
\varphi_e(y) = \varphi_{f(e)}(y) = g(e, y).
\]
This concludes the proof. \( \square \)

Before showing that statement (1) is true (and hence (2) as well), consider how bizarre it is. Think of \( e \) as being a computer program; statement (1) says that given any partial computable \( g(x, y) \), you can find a computer program \( e \) that computes \( g_e(y) \simeq g(e, y) \). In other words, you can find a computer program that computes a function that references the program itself.

**Theorem thy.2.** The two statements in Lemma thy.1 are true. Specifically, for every partial computable function \( g(x, y) \), there is an index \( e \) such that for every \( y \),
\[
\varphi_e(y) \simeq g(e, y).
\]

**Proof.** The ingredients are already implicit in the discussion of the halting problem above. Let \( \text{diag}(x) \) be a computable function which for each \( x \) returns an index for the function \( f_x(y) \simeq \varphi_x(x, y) \), i.e.
\[
\varphi_{\text{diag}(x)}(y) \simeq \varphi_x(x, y).
\]

Think of \( \text{diag} \) as a function that transforms a program for a 2-ary function into a program for a 1-ary function, obtained by fixing the original program as its first argument. The function \( \text{diag} \) can be defined formally as follows: first define \( s \) by
\[
s(x, y) \simeq \text{Un}^2(x, x, y),
\]
where \( \text{Un}^2 \) is a 3-ary function that is universal for partial computable 2-ary functions. Then, by the \( s\)-\( m\)-\( n \) theorem, we can find a primitive recursive function \( \text{diag} \) satisfying
\[
\varphi_{\text{diag}(x)}(y) \simeq s(x, y).
\]

Now, define the function \( l \) by
\[
l(x, y) \simeq g(\text{diag}(x), y).
\]
and let \( "l" \) be an index for \( l \). Finally, let \( e = \text{diag}("l") \). Then for every \( y \), we have
\[
\varphi_e(y) \simeq \varphi_{\text{diag}("l")}(y) \simeq \varphi_{l(\"l\"), y} \simeq l(\"l\"), y) \simeq g(\text{diag}(\"l\"), y) \simeq g(e, y),
\]
as required. \( \square \)
What’s going on? Suppose you are given the task of writing a computer
program that prints itself out. Suppose further, however, that you are working
with a programming language with a rich and bizarre library of string functions.
In particular, suppose your programming language has a function diag which
works as follows: given an input string $s$, diag locates each instance of the
symbol ‘x’ occurring in $s$, and replaces it by a quoted version of the original
string. For example, given the string

$$\text{hello x world}$$

as input, the function returns

$$\text{hello ’hello x world’ world}$$

as output. In that case, it is easy to write the desired program; you can check
that

$$\text{print(diag(’print(diag(x))’))}$$

does the trick. For more common programming languages like C++ and Java,
the same idea (with a more involved implementation) still works.

We are only a couple of steps away from the proof of the fixed-point theo-
rem. Suppose a variant of the print function $\text{print}(x, y)$ accepts a string $x$ and
another numeric argument $y$, and prints the string $x$ repeatedly, $y$ times. Then
the “program”

$$\text{getinput}(y); \text{print(diag(’getinput(y); print(diag(x), y)’)}, y)$$

prints itself out $y$ times, on input $y$. Replacing the $\text{getinput—print—diag}$
skeleton by an arbitrary function $g(x, y)$ yields

$$g(\text{diag(’g(diag(x), y)’)}, y)$$

which is a program that, on input $y$, runs $g$ on the program itself and $y$.
Thinking of “quoting” with “using an index for,” we have the proof above.

For now, it is o.k. if you want to think of the proof as formal trickery, or
black magic. But you should be able to reconstruct the details of the argument
given above. When we prove the incompleteness theorems (and the related
“fixed-point theorem”) we will discuss other ways of understanding why it
works.

The same idea can be used to get a “fixed point” combinator. Suppose you
digression

have a lambda term $g$, and you want another term $k$ with the property that $k$
is $\beta$-equivalent to $gk$. Define terms

$$\text{diag}(x) = xx$$

and

$$l(x) = g(\text{diag}(x))$$
using our notational conventions; in other words, $l$ is the term $\lambda x. g(xx)$. Let $k$ be the term $ll$. Then we have

$$k = (\lambda x. g(xx))(\lambda x. g(xx))$$
$$\rightarrow g((\lambda x. g(xx))(\lambda x. g(xx)))$$
$$= gk.$$

If one takes

$$Y = \lambda g. ((\lambda x. g(xx))(\lambda x. g(xx)))$$

then $Yg$ and $g(Yg)$ reduce to a common term; so $Yg \equiv g(Yg)$. This is known as “Curry’s combinator.” If instead one takes

$$Y = (\lambda x g. g(xx))(\lambda x g. g(xx))$$

then in fact $Yg$ reduces to $g(Yg)$, which is a stronger statement. This latter version of $Y$ is known as “Turing’s combinator.”

Photo Credits

Bibliography