The following gives a number of important equivalent statements of what it means to be computably enumerable.

**Theorem thy.1.** Let $S$ be a set of natural numbers. Then the following are equivalent:

1. $S$ is computably enumerable.
2. $S$ is the range of a partial computable function.
3. $S$ is empty or the range of a primitive recursive function.
4. $S$ is the domain of a partial computable function.

The first three clauses say that we can equivalently take any non-empty computably enumerable set to be enumerated by either a computable function, a partial computable function, or a primitive recursive function. The fourth clause tells us that if $S$ is computably enumerable, then for some index $e$,

$$S = \{x : \varphi_e(x) \downarrow\}.$$

In other words, $S$ is the set of inputs on for which the computation of $\varphi_e$ halts. For that reason, computably enumerable sets are sometimes called *semi-decidable*: if a number is in the set, you eventually get a “yes,” but if it isn’t, you never get a “no”!

**Proof.** Since every primitive recursive function is computable and every computable function is partial computable, (3) implies (1) and (1) implies (2). (Note that if $S$ is empty, $S$ is the range of the partial computable function that is nowhere defined.) If we show that (2) implies (3), we will have shown the first three clauses equivalent.

So, suppose $S$ is the range of the partial computable function $\varphi_e$. If $S$ is empty, we are done. Otherwise, let $a$ be any element of $S$. By Kleene’s normal form theorem, we can write

$$\varphi_e(x) = U(\mu s T(e, x, s)).$$

In particular, $\varphi_e(x) \downarrow = y$ if and only if there is an $s$ such that $T(e, x, s)$ and $U(s) = y$. Define $f(z)$ by

$$f(z) = \begin{cases} U((z)_1) & \text{if } T(e, (z)_0, (z)_1) \\ a & \text{otherwise.} \end{cases}$$

Then $f$ is primitive recursive, because $T$ and $U$ are. Expressed in terms of Turing machines, if $z$ codes a pair $((z)_0, (z)_1)$ such that $(z)_1$ is a halting computation of machine $e$ on input $(z)_0$, then $f$ returns the output of the computation; otherwise, it returns $a$. We need to show that $S$ is the range of $f$, i.e.,
for any natural number \( y, y \in S \) if and only if it is in the range of \( f \). In the forwards direction, suppose \( y \in S \). Then \( y \) is in the range of \( \varphi_e \), so for some \( x \) and \( s \), \( T(e, x, s) \) and \( U(s) = y \); but then \( y = f((x, s)) \). Conversely, suppose \( y \) is in the range of \( f \). Then either \( y = a \), or for some \( z \), \( T(e, (z)_0, (z)_1) \) and \( U((z)_1) = y \). Since, in the latter case, \( \varphi_e(x) \downarrow = y \), either way, \( y \) is in \( S \).

(The notation \( \varphi_e(x) \downarrow = y \) means “\( \varphi_e(x) \) is defined and equal to \( y \).” We could just as well use \( \varphi_e(x) = y \), but the extra arrow is sometimes helpful in reminding us that we are dealing with a partial function.)

To finish up the proof of Theorem thy.1, it suffices to show that (1) and (4) are equivalent. First, let us show that (1) implies (4). Suppose \( S \) is the range of a computable function \( f \), i.e.,

\[
S = \{ y : \text{for some } x, f(x) = y \}.
\]

Let

\[
g(y) = \mu x \ f(x) = y.
\]

Then \( g \) is a partial computable function, and \( g(y) \) is defined if and only if for some \( x, f(x) = y \). In other words, the domain of \( g \) is the range of \( f \). Expressed in terms of Turing machines: given a Turing machine \( F \) that enumerates the elements of \( S \), let \( G \) be the Turing machine that semi-decides \( S \) by searching through the outputs of \( F \) to see if a given element is in the set.

Finally, to show (4) implies (1), suppose that \( S \) is the domain of the partial computable function \( \varphi_e \), i.e.,

\[
S = \{ x : \varphi_e(x) \downarrow \}.
\]

If \( S \) is empty, we are done; otherwise, let \( a \) be any element of \( S \). Define \( f \) by

\[
f(z) = \begin{cases} (z)_0 & \text{if } T(e, (z)_0, (z)_1) \\ a & \text{otherwise.} \end{cases}
\]

Then, as above, a number \( x \) is in the range of \( f \) if and only if \( \varphi_e(x) \downarrow \), i.e., if and only if \( x \in S \). Expressed in terms of Turing machines: given a machine \( M_e \) that semi-decides \( S \), enumerate the elements of \( S \) by running through all possible Turing machine computations, and returning the inputs that correspond to halting computations.

The fourth clause of Theorem thy.1 provides us with a convenient way of enumerating the computably enumerable sets: for each \( e \), let \( W_e \) denote the domain of \( \varphi_e \). Then if \( A \) is any computably enumerable set, \( A = W_e \), for some \( e \).

The following provides yet another characterization of the computably enumerable sets.

**Theorem thy.2.** A set \( S \) is computably enumerable if and only if there is a computable relation \( R(x, y) \) such that

\[
S = \{ x : \exists y \ R(x, y) \}.
\]
Proof. In the forward direction, suppose $S$ is computably enumerable. Then for some $e$, $S = W_e$. For this value of $e$ we can write $S$ as

$$S = \{ x : \exists y T(e, x, y) \}.$$ 

In the reverse direction, suppose $S = \{ x : \exists y R(x, y) \}$. Define $f$ by

$$f(x) \simeq \mu y \text{ AtomRx}, y.$$ 

Then $f$ is partial computable, and $S$ is the domain of $f$. \qed

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Bibliography