Defining Functions using Self-Reference

It is generally useful to be able to define functions in terms of themselves. For example, given computable functions $k$, $l$, and $m$, the fixed-point lemma tells us that there is a partial computable function $f$ satisfying the following equation for every $y$:

\[
f(y) \simeq \begin{cases} 
k(y) & \text{if } l(y) = 0 \\
f(m(y)) & \text{otherwise.}
\end{cases}
\]

Again, more specifically, $f$ is obtained by letting

\[
g(x, y) \simeq \begin{cases} 
k(y) & \text{if } l(y) = 0 \\
\varphi_{\beta}(m(y)) & \text{otherwise}
\end{cases}
\]

and then using the fixed-point lemma to find an index $e$ such that $\varphi_{\beta}(y) = g(e, y)$.

For a concrete example, the “greatest common divisor” function $\text{gcd}(u, v)$ can be defined by

\[
\text{gcd}(u, v) \simeq \begin{cases} 
v & \text{if } 0 = 0 \\
\text{gcd}(\text{mod}(v, u), u) & \text{otherwise}
\end{cases}
\]

where $\text{mod}(v, u)$ denotes the remainder of dividing $v$ by $u$. An appeal to the fixed-point lemma shows that $\text{gcd}$ is partial computable. (In fact, this can be put in the format above, letting $y$ code the pair $\langle u, v \rangle$.) A subsequent induction on $u$ then shows that, in fact, $\text{gcd}$ is total.

Of course, one can cook up self-referential definitions that are much fancier than the examples just discussed. Most programming languages support definitions of functions in terms of themselves, one way or another. Note that this is a little bit less dramatic than being able to define a function in terms of an index for an algorithm computing the functions, which is what, in full generality, the fixed-point theorem lets you do.

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Bibliography