thy.1 Complete Computably Enumerable Sets

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Definition thy.1. A set A is a *complete computably enumerable set* (under many-one reducibility) if

1. A is computably enumerable, and
2. for any other computably enumerable set B, $B \leq_m A$.

In other words, complete computably enumerable sets are the "hardest" computably enumerable sets possible; they allow one to answer questions about *any* computably enumerable set.

Theorem thy.2. *K, K_0, and K_1 are all complete computably enumerable sets.*

Proof. To see that K_0 is complete, let B be any computably enumerable set. Then for some index e,

$$B = W_e = \{x : \varphi_e(x) \downarrow\}.$$

Let f be the function $f(x) = \langle e, x \rangle$. Then for every natural number x, $x \in B$ if and only if $f(x) \in K_0$. In other words, f reduces B to K_0.

To see that K_1 is complete, note that in the proof of **??** we reduced K_0 to it. So, by **??**, any computably enumerable set can be reduced to K_1 as well.

K can be reduced to K_0 in much the same way. □

Problem thy.1. Give a reduction of K to K_0.

So, it turns out that all the examples of computably enumerable sets that we have considered so far are either computable, or complete. This should seem strange! Are there any examples of computably enumerable sets that are neither computable nor complete? The answer is yes, but it wasn't until the middle of the 1950s that this was established by Friedberg and Muchnik, independently. digression

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Bibliography