Complete Computably Enumerable Sets

**Definition thy.1.** A set $A$ is a *complete computably enumerable set* (under many-one reducibility) if

1. $A$ is computably enumerable, and
2. for any other computably enumerable set $B$, $B \leq_m A$.

In other words, complete computably enumerable sets are the “hardest” computably enumerable sets possible; they allow one to answer questions about any computably enumerable set.

**Theorem thy.2.** $K$, $K_0$, and $K_1$ are all complete computably enumerable sets.

**Proof.** To see that $K_0$ is complete, let $B$ be any computably enumerable set. Then for some index $e$,

$$B = W_e = \{x : \varphi_e(x) \downarrow\}.$$  

Let $f$ be the function $f(x) = \langle e, x \rangle$. Then for every natural number $x$, $x \in B$ if and only if $f(x) \in K_0$. In other words, $f$ reduces $B$ to $K_0$.

To see that $K_1$ is complete, note that in the proof of ?? we reduced $K_0$ to it. So, by ??, any computably enumerable set can be reduced to $K_1$ as well. $K$ can be reduced to $K_0$ in much the same way.

**Problem thy.1.** Give a reduction of $K$ to $K_0$.

So, it turns out that all the examples of computably enumerable sets that we have considered so far are either computable, or complete. This should seem strange! Are there any examples of computably enumerable sets that are neither computable nor complete? The answer is yes, but it wasn’t until the middle of the 1950s that this was established by Friedberg and Muchnik, independently.

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Bibliography